New families of wrapped distributions for modeling skew circular data

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Abstract: We discuss circular distributions obtained by wrapping the classical exponential and Laplace distributions on the real line around the circle. We present explicit forms for their densities and distribution functions, as well as their trigonometric moments and related parameters, and discuss main properties of these laws. Both distributions are very promising as models for directional data which are asymmetric.

Keywords: Angular data; asymmetric Laplace distribution; bird orientations; circular data; double exponential distribution; wrapped distribution.

1 Introduction

Let X be a real random variable (r.v.) with probability density function (p.d.f.) f and characteristic function (ch.f.) ϕ. Then the corresponding wrapped r.v.

$$X_w = X (\text{mod } 2\pi) \tag{1}$$

has density

$$f_w(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi), \;\; \theta \in [0, 2\pi), \tag{2}$$

and the characteristic function (the discrete Fourier transform)

$$\phi_p = Ee^{ipX_w} = \phi(p), \;\; p = 0, \pm 1, \pm 2, \ldots \tag{3}$$

(See, e.g., [8, 13]).

The cases of wrapped Cauchy, normal, and stable distributions have been studied extensively (see, e.g., [4, 12]). In this work we present basic theory of *wrapped exponential and double-exponential* (Laplace) distributions, which were introduced recently in [6, 7].

The exponential distribution, with its important memoryless property, provides a standard model in diverse fields such as reliability, queuing theory, and others (see, e.g., [1]). The classical Laplace distribution and its skew generalizations are competing with normal and other distributions in stochastic modeling, particularly in financial applications (see, e.g., [11] and references therein). We believe that their circular analogs can find interesting applications in directional data, when they resemble the characteristic shape of the Laplace distribution (with its possible skewness and sharp peak at the mode). Such data frequently results from orientation experiments in biology (see, e.g., [2, 3, 5, 14, 15]).

Below we sketch a theory of wrapped exponential and Laplace distributions. First, we define them in Section 2, where we also present their basic properties. Then, we summarize their important properties in Section 3. More detailed information with estimation, applications, and proofs can be found in [6, 7].

2 Definitions and basic properties

When we apply (2) and (3) to the exponential distribution with p.d.f. $f(x) = \lambda e^{-\lambda x}$, $x > 0$, we obtain a wrapped exponential distribution, denoted by $WE(\lambda)$ with $\lambda > 0$. The following Table 1 provides the characteristic function (ch.f.), density function (p.d.f.), and cumulative distribution function (c.d.f.) of the wrapped exponential distribution $WE(\lambda)$, $\lambda \in R$.

ch.f.	$\phi_p = \frac{1}{1-ip/\lambda}$, $p = 0, \pm 1, \pm 2, \ldots$
p.d.f.	$f(\theta) = \frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}$, $\theta \in [0, 2\pi)$
c.d.f.	$F(\theta) = \frac{1-e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}$, $\theta \in [0, 2\pi)$

Table 1: The wrapped exponential distribution $WE(\lambda)$.

The p.d.f. should be extended in a periodic fashion for the values of θ outside of the interval $[0, 2\pi)$. The case $\lambda = 0$ is defined by a continuous extension and corresponds to the circular uniform distribution (when $\lambda \to 0^+$ then the c.d.f. in Table 1 converges to $\frac{\theta}{2\pi}$). The case $\lambda < 0$ results from wrapping the "negative" exponential distribution with parameter $|\lambda| > 0$, whose p.d.f. is $f(x) = |\lambda| e^{|\lambda| x}$, $x < 0$. Moreover, we have the relation

$$\Theta \sim WE(\lambda) \text{ if and only if } 2\pi - \Theta \sim WE(-\lambda). \tag{4}$$

Remark. We note a very interesting and curious property: The restriction of the linear exponential r.v. X to the interval $[0, 2\pi)$ (that is $X|X \leq 2\pi$) has the same distribution as the wrapped r.v. X_w given by (1).

Consider now an asymmetric Laplace r.v. X with the density function

$$f(x) = \lambda(1/\kappa + \kappa)^{-1} \begin{cases} e^{-\lambda\kappa|x|}, & \text{for } x \geq 0, \\ e^{-(\lambda/\kappa)|x|}, & \text{for } x < 0, \end{cases} \tag{5}$$

and the ch.f.

$$\phi(t) = \frac{1}{1 + t^2/\lambda^2 - it(1/\kappa - \kappa)/\lambda} = \frac{1}{1 - it/(\lambda\kappa)} \frac{1}{1 + it/(\lambda/\kappa)}, \quad t \in R, \tag{6}$$

where $\lambda, \kappa > 0$; see [11] for theory and applications of these distributions and their various generalizations. (For $\kappa = 1$ we obtain the classical (symmetric) Laplace density.) When we apply (2) and (3) to the above distribution, we obtain a *wrapped Laplace distribution*, denoted by $WL(\lambda, \kappa)$. This is defined in Table 2 below, which provides the characteristic function (ch.f.), density function (p.d.f.), and cumulative distribution function (c.d.f.) of the wrapped Laplace distribution $WL(\lambda, \kappa)$, $\lambda \in R, \kappa > 0$.

ch.f.	$\phi_p = \frac{1}{1-ip/(\lambda\kappa)} \frac{1}{1+ip/(\lambda/\kappa)}, \quad p = 0, \pm 1, \pm 2, \ldots$
p.d.f.	$f_w(\theta) = \frac{\lambda\kappa}{1+\kappa^2} \left(\frac{e^{-\lambda\kappa\theta}}{1-e^{-2\pi\lambda\kappa}} + \frac{e^{(\lambda/\kappa)\theta}}{e^{2\pi\lambda/\kappa}-1} \right), \quad \theta \in [0, 2\pi)$
c.d.f.	$F_w(\theta) = \frac{1}{1+\kappa^2} \frac{1-e^{-\lambda\kappa\theta}}{1-e^{-2\pi\lambda\kappa}} + \frac{\kappa^2}{1+\kappa^2} \frac{e^{(\lambda/\kappa)\theta}-1}{1-e^{-2\pi\lambda/\kappa}}, \quad \theta \in [0, 2\pi)$

Table 2: The wrapped Laplace distribution $WL(\lambda, \kappa)$.

The p.d.f. should be extended in a periodic fashion for the values of θ outside of the interval $[0, 2\pi)$. Observe that the p.d.f. in Table 2 integrates to 1 on $[0, 2\pi)$ for any values of λ, including $\lambda < 0$, and we have

$$WL(-\lambda, \kappa) = WL(\lambda, 1/\kappa). \tag{7}$$

It is also easy to see that

$$\Theta \sim WL(\lambda, \kappa) \text{ if and only if } 2\pi - \Theta \sim WE(\lambda, 1/\kappa).$$

In case of a symmetric Laplace distribution with $\kappa = 1$, the wrapped Laplace density simplifies to

$$\frac{\lambda}{2}\frac{e^{(2\pi-\theta)\lambda} + e^{\lambda\theta}}{e^{2\pi\lambda} - 1}, \tag{8}$$

and we have $2\pi - \Theta \stackrel{d}{=} \Theta$ in this case, where $\stackrel{d}{=}$ denotes distributional equivalence.

Remark. The Laplace distribution is a mixture of a positive and a negative exponential distributions, since f in (5) can be written as

$$f(x) = pf_1(x) + (1-p)f_2(x), \quad x \in R, \tag{9}$$

where

$$f_1(x) = \lambda_1 e^{-\lambda_1 x} \ (x > 0), \quad f_2(x) = \lambda_2 e^{\lambda_2 x} \ (x < 0), \tag{10}$$

and

$$p = \frac{1}{\kappa^2 + 1}, \quad \lambda_1 = \lambda\kappa, \quad \lambda_2 = \lambda/\kappa. \tag{11}$$

Consequently, the wrapped Laplace distribution is a mixture of two wrapped exponential distribution, as the p.d.f. in Table 2 can be written as

$$f_w(\theta) = p\frac{\lambda_1 e^{-\lambda_1\theta}}{1 - e^{-2\pi\lambda_1}} + (1-p)\frac{\lambda_2 e^{\lambda_2\theta}}{e^{2\pi\lambda_2} - 1}, \quad \theta \in [0, 2\pi), \tag{12}$$

with the parameters as in (11). The corresponding wrapped Laplace $WL(\lambda, \kappa)$ r.v. Θ admits the representation

$$\Theta \stackrel{d}{=} I\Theta_1 + (1-I)\Theta_2, \tag{13}$$

where Θ_1 and Θ_2 are independent wrapped exponential $WE(\lambda\kappa)$ and $WE(-\lambda/\kappa)$ r.v.'s, and I is an indicator random variable (independent of Θ_1 and Θ_2) taking on the values 1 and 0 with probabilities $1/(1+\kappa^2)$ and $\kappa^2/(1+\kappa^2)$, respectively.

Remark. By the factorization of the asymmetric Laplace ch.f., the corresponding r.v. has the same distribution as the difference of two independent exponential random variables (see, e.g., [11]). Since the wrapped Laplace ch.f. given in Table 2 admits a similar factorization, we obtain an analogous representation for the wrapped Laplace r.v. $\Theta \sim WL(\lambda, \kappa)$ viz.

$$\Theta \stackrel{d}{=} \Theta_1 + \Theta_2 \,(\text{mod } 2\pi). \tag{14}$$

where, Θ_1 and Θ_2 are independent wrapped exponential $WE(\lambda\kappa)$ and $WE(-\lambda/\kappa)$ r.v.'s, mentioned before.

Remark. The $WL(\lambda, \kappa)$ distribution converges weakly to the circular uniform distribution as $\lambda \to 0$, or $\kappa \to 0^+$, or $\kappa \to \infty$.

Remark. A three-parameter class of distributions can be defined by introducing a location parameter $\eta \in [0, 2\pi)$ and shifting the wrapped Laplace p.d.f. (defined on R by a periodic extension) by η with the resulting densities of the form

$$g(\theta) = f_w(\theta - \eta)$$

with f_w given in Table 2. Parameter η clearly corresponds to the mode for such a family.

2.1 Characterization of the densities

Note that the p.d.f. of the wrapped exponential distribution $WE(\lambda)$ is strictly decreasing on the interval $[0, 2\pi)$ if $\lambda > 0$, and strictly increasing on $[0, 2\pi)$ if $\lambda < 0$.

In case of the wrapped Laplace distributions, the densities are unimodal with the sharp peak at the mode. Basic properties of the densities are described in the following result taken from [7].

Proposition 2.1 *Let $f(\cdot; \lambda, \kappa)$ be the density of the $WL(\lambda, \kappa)$ distribution with $\lambda > 0$. Then*
(i) $f(\cdot; \lambda, \kappa)$ is strictly decreasing on $(0, \theta^*)$ and strictly increasing on $(\theta^*, 2\pi)$, where

$$\theta^* = \left(\frac{\lambda}{\kappa} + \lambda\kappa\right)^{-1} \ln\left(\kappa^2 \frac{e^{2\pi\lambda/\kappa} - 1}{1 - e^{-2\pi\lambda\kappa}}\right). \tag{15}$$

Moreover

$$\theta^* > \pi \text{ for } \kappa < 1, \quad \theta^* = \pi \text{ for } \kappa = 1, \quad \text{and } \theta^* < \pi \text{ for } \kappa > 1; \tag{16}$$

(ii) The maximum and the minimum values of $f(\cdot; \lambda, \kappa)$ are

$$f(0; \lambda, \kappa) = \lim_{\theta \to 2\pi} f(\theta; \lambda, \kappa) = \frac{\lambda\kappa}{1 + \kappa^2}\left(\frac{1}{e^{2\pi\lambda/\kappa} - 1} + \frac{1}{1 - e^{-2\pi\lambda\kappa}}\right) \tag{17}$$

and

$$f(\theta^*;\lambda,\kappa) = e^{\frac{2\pi\lambda\kappa}{1+\kappa^2}} \left(\frac{e^{2\pi\lambda/\kappa}-1}{2\pi\lambda/\kappa}\right)^{\frac{-\kappa^2}{1+\kappa^2}} \left(\frac{e^{2\pi\lambda\kappa}-1}{2\pi\lambda\kappa}\right)^{\frac{-1}{1+\kappa^2}}, \tag{18}$$

respectively.
(iii) For any given $\lambda \geq 0$ *and* $\kappa \geq 0$, *we have*

$$f(\theta;\lambda,1/\kappa) = f(2\pi-\theta;\lambda,\kappa), \;\; \theta \in [0,2\pi). \tag{19}$$

2.2 Trigonometric moments and related parameters

Computation of the trigonometric moments and related parameters of the wrapped exponential distribution is straightforward. For the convenience of the reader we summarize some common parameters in Table 3. Note that since the ϕ_p's are the Fourier coefficients, we have the following Fourier representation of the wrapped exponential density:

$$f(\theta) = \sum_p \phi_p e^{-ip\theta} = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty} \frac{\lambda^2}{\lambda^2+p^2}\cos p\theta + \frac{\lambda p}{\lambda^2+p^2}\sin p\theta\right]. \tag{20}$$

Similarly the trigonometric moments of the wrapped Laplace distribution are also easy to derive, utilizing the mixture representation (13) and formulas for the trigonometric moments of the wrapped exponential distribution given in Table 3. Here, the density of the $WL(\lambda,\kappa)$ distribution admits the Fourier representation

$$f(\theta;\lambda,\kappa) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty} \frac{\kappa\lambda[\kappa\lambda(p^2+\lambda^2)\cos p\theta + p\lambda^2(1-\kappa^2)\sin p\theta]}{(\lambda^2\kappa^2+p^2)(\kappa^2p^2+\lambda^2)}\right]. \tag{21}$$

Other common parameters of the wrapped Laplace laws can be obtained with ease, perhaps with the exception of the circular median. They are summarized in Table 4. Note that the mean direction lies in the interval $[0,\pi/2)$ for $\kappa \leq 1$ and in the interval $[3\pi/2, 2\pi)$ for $\kappa > 1$. This restriction is caused by the fact that the mode is equal to zero.

A median direction ξ_0 of a circular distribution with density f is any solution (in the interval $[0,2\pi)$) of

$$\int_{\xi_0}^{\xi_0+\pi} f(\theta)d\theta = \int_{\xi_0+\pi}^{\xi_0+2\pi} f(\theta)d\theta = \frac{1}{2}, \tag{22}$$

where the density f satisfies

$$f(\xi_0) > f(\xi_0+\pi), \tag{23}$$

Parameter	Definition	Value
Trig. moments	$\phi_p = \alpha_p + i\beta_p$ $p = 0, \pm 1, \pm 2, \dots$	$\alpha_p = \frac{\lambda^2}{\lambda^2+p^2}$ $\beta_p = \frac{p\lambda}{\lambda^2+p^2}$
Central trig. moments	$\bar{\alpha}_p = \rho_p \cos(\mu_p^0 - p\mu_0)$ $\bar{\beta}_p = \rho_p \sin(\mu_p^0 - p\mu_0)$	$\bar{\alpha}_p = \frac{\lvert\lambda\rvert}{\sqrt{\lambda^2+p^2}} \cos\left(\tan^{-1}\frac{p}{\lambda} - p\tan^{-1}\frac{1}{\lambda}\right)$ $\bar{\beta}_p = \frac{\lvert\lambda\rvert}{\sqrt{\lambda^2+p^2}} \sin\left(\tan^{-1}\frac{p}{\lambda} - p\tan^{-1}\frac{1}{\lambda}\right)$
ρ_p and μ_p^0	$\phi_p = \rho_p e^{i\mu_p^0}, \ \ p \geq 0,$	$\rho_p = \frac{\lvert\lambda\rvert}{\sqrt{\lambda^2+p^2}}$ $\mu_p^0 = \begin{cases} \tan^{-1}\frac{p}{\lambda}, & \lambda > 0 \\ 2\pi + \tan^{-1}\frac{p}{\lambda}, & \lambda < 0 \end{cases}$
Resultant length	$\rho = \rho_1$	$\frac{\lvert\lambda\rvert}{\sqrt{\lambda^2+1}}$
Mean direction	$\mu_0 = \mu_1^0$	$\tan^{-1}\frac{1}{\lambda}, \quad \lambda > 0$ $2\pi + \tan^{-1}\frac{1}{\lambda}, \quad \lambda < 0$
Circular variance	$V_0 = 1 - \rho$	$\frac{\sqrt{1+\lambda^2} - \lvert\lambda\rvert}{\sqrt{1+\lambda^2}}$
Circular standard deviation	$\sigma_0 = \sqrt{-2\ln(1-V_0)}$	$\sqrt{\ln(1+1/\lambda^2)}$
Skewness	$\gamma_1^0 = \bar{\beta}_2 / V_0^{3/2}$	$\frac{-2\lambda}{(1+\lambda^2)^{1/4}(4+\lambda^2)(\sqrt{1+\lambda^2}-\lambda)^{3/2}}$
Kurtosis	$\gamma_2^0 = \frac{\bar{\alpha}_2 - (1-V_0)^4}{V_0^2}$	$\frac{3\lambda^2}{(1+\lambda^2)(4+\lambda^2)(\sqrt{1+\lambda^2}-\lvert\lambda\rvert)^2}$
Median	See (22)-(23)	$\frac{1}{\lambda}\ln\frac{2}{1+e^{-\lambda\pi}} + \begin{cases} 0, & \lambda > 0 \\ \pi, & \lambda < 0 \end{cases}$

Table 3: Trigonometric moments and related parameters of the wrapped exponential distribution $WE(\lambda)$.

see, e.g., [13]. Clearly, the median direction of the wrapped symmetric Laplace distribution $WL(\lambda, 1)$ is equal to zero (and coincides with the mean direction and the mode). In the following result, proved in [7], we describe the median direction for the general case.

Proposition 2.2 *Let $\Theta \sim WL(\lambda, \kappa)$. Then, Θ admits a unique median direction given by*

$$\xi_0 = \begin{cases} \xi^*, & \text{for } \lambda > 0, \ 0 < \kappa < 1, \\ \xi^* + \pi, & \text{for } \lambda > 0, \ \kappa > 1, \end{cases} \tag{24}$$

where $\xi^* \in [0, \pi]$ is the unique solution of the equation

$$\frac{1}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\xi}}{1+e^{-\lambda\kappa\pi}} + \kappa^2 \frac{e^{\lambda\xi/\kappa}}{1+e^{\lambda\pi/\kappa}}\right) = \frac{1}{2}. \tag{25}$$

Clearly, the median is less than π for $\kappa < 1$ and greater than π for $\kappa > 1$.

Parameter(s)	**Value(s)**	**Case** $\kappa = 1$
Trig. moments	$\alpha_p = \frac{\kappa^2\lambda^2(p^2+\lambda^2)}{(\lambda^2\kappa^2+p^2)(p^2\kappa^2+\lambda^2)}$ $\beta_p = \frac{p\kappa\lambda^3(1-\kappa^2)}{(\lambda^2\kappa^2+p^2)(p^2\kappa^2+\lambda^2)}$	$\alpha_p = \frac{\lambda^2}{p^2+\lambda^2}$ $\beta_p = 0$
Central trig. moments	$\bar{\alpha}_p = \rho_p \cos(\mu_p^0 - p\mu_0)$ $\bar{\beta}_p = \rho_p \sin(\mu_p^0 - p\mu_0)$	$\bar{\alpha}_p = \frac{\lambda^2}{\lambda^2+p^2}$ $\bar{\beta}_p = 0$
ρ_p and μ_p^0	$\rho_p = \frac{\lambda^2}{\sqrt{\lambda^2\kappa^2+p^2}\sqrt{\lambda^2/\kappa^2+p^2}}, \quad p \geq 0$ $\mu_p^0 = \begin{cases} \tan^{-1}\frac{p}{\lambda\kappa} - \tan^{-1}\frac{p\kappa}{\lambda}, & \kappa \leq 1 \\ 2\pi + \tan^{-1}\frac{p}{\lambda\kappa} - \tan^{-1}\frac{p\kappa}{\lambda}, & \kappa > 1 \end{cases}$	$\rho_p = \frac{\lambda^2}{\lambda^2+p^2}$ $\mu_p^0 = 0$
Resultant length	$\rho = \frac{\lambda^2}{\sqrt{1+\lambda^2\kappa^2}\sqrt{\lambda^2/\kappa^2+1}}$	$\rho = \frac{\lambda^2}{\lambda^2+1}$
Mean direction	$\mu_0 = \begin{cases} \tan^{-1}\frac{1}{\lambda\kappa} - \tan^{-1}\frac{\kappa}{\lambda}, & \kappa \leq 1 \\ 2\pi + \tan^{-1}\frac{1}{\lambda\kappa} - \tan^{-1}\frac{\kappa}{\lambda}, & \kappa > 1 \end{cases}$	$\mu_0 = 0$
Circular variance	$V_0 = 1 - \frac{\lambda^2}{\sqrt{1+\lambda^2\kappa^2}\sqrt{1+\lambda^2/\kappa^2}}$	$V_0 = \frac{1}{1+\lambda^2}$
Circular standard deviation	$\sigma_0 = \sqrt{\ln\left(\kappa^2 + \frac{1}{\lambda^2}\right) + \ln\left(\frac{1}{\kappa^2} + \frac{1}{\lambda^2}\right)}$	$\sigma_0 = \sqrt{2\ln\left(1 + \frac{1}{\lambda^2}\right)}$
Skewness	$\gamma_1^0 = \bar{\beta}_2/V_0^{3/2}$, where $\bar{\beta}_2 = \frac{2\lambda^3\kappa(\lambda^2\kappa^2+3)-2(\lambda^3/\kappa)(\lambda^2/\kappa^2+3)}{(1+\lambda^2\kappa^2)(4+\lambda^2\kappa^2)(1+\lambda^2/\kappa^2)(4+\lambda^2/\kappa^2)}$	$\gamma_1^0 = 0$
Kurtosis	$\gamma_2^0 = \frac{\bar{\alpha}_2-(1-V_0)^4}{V_0^2}$, where $\hat{\alpha}_2 = \frac{\lambda^4(\lambda^2\kappa^2+3)(\lambda^2/\kappa^2+3)+4\lambda^4}{(1+\lambda^2\kappa^2)(4+\lambda^2\kappa^2)(1+\lambda^2/\kappa^2)(4+\lambda^2/\kappa^2)}$	$\gamma_2^0 = (1+\lambda^2)\lambda^2 \times$ $\times\left(\frac{1}{4+\lambda^2} - \frac{\lambda^6}{(1+\lambda^2)^2}\right)$
Median	$\xi_0 = \begin{cases} \xi^*, & \lambda > 0, 0 < \kappa < 1 \\ \xi^* + \pi, & \lambda > 0, \kappa > 1, \end{cases}$ where $\frac{1}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\xi^*}}{1+e^{-\lambda\kappa\pi}} + \frac{\kappa^2 e^{\lambda\xi^*/\kappa}}{1+e^{\lambda\pi/\kappa}}\right) = \frac{1}{2}$, $\xi^* \in [0, \pi]$	$\xi_0 = 0$

Table 4: Trigonometric moments and related parameters of the wrapped Laplace distribution $WL(\lambda, \kappa)$. For the definitions see Table 1. The entries in the last column correspond to the symmetric case with $\kappa = 1$.

3 Some important properties

In this section we show that wrapped exponential and Laplace distributions share some of the well-known properties of their analogs on the real line.

3.1 Infinite divisibility

An angular r.v. Θ (and its probability distribution) is infinitely divisible if for any integer $n \geq 1$ there exist i.i.d. angular r.v.'s $\Theta_1, \ldots, \Theta_n$ such that

$$\Theta_1 + \cdots + \Theta_n (\text{mod } 2\pi) \stackrel{d}{=} \Theta. \tag{26}$$

Since a circular variable obtained by wrapping an infinitely divisible random variable is infinitely divisible (see [13]), the infinite divisibility of the wrapped exponential and Laplace distributions follow from that of the linear exponential and Laplace distributions.

Proposition 3.1 *If $\Theta \sim WE(\lambda)$ with $\lambda \in R$, then Θ is infinitely divisible. Moreover, for any positive integer $n \geq 1$ the equality in distribution (26) holds where the Θ_i's have the uniform circular distribution for $\lambda = 0$ and the wrapped gamma distribution with the ch.f.*

$$\phi_p = \left(\frac{1}{1 - ip/\lambda} \right)^{1/n} \tag{27}$$

for $\lambda \neq 0$.

Proposition 3.2 *If $\Theta \sim WL(\lambda, \kappa)$, where $\lambda, \kappa \geq 0$, then Θ is infinitely divisible. Moreover, for any positive integer $n \geq 1$ the equality in distribution (26) holds with uniform circular variable Θ_1 if either $\lambda = 0$ or $\kappa = 0$, and otherwise with*

$$\Theta_1 \stackrel{d}{=} \Theta' + \Theta'' \, (mod \; 2\pi), \tag{28}$$

where Θ' and Θ'' are independent wrapped gamma r.v.'s with ch.f.'s

$$\left(\frac{1}{1 - ip/(\lambda\kappa)} \right)^{1/n} \quad and \quad \left(\frac{1}{1 + ip/(\lambda/\kappa)} \right)^{1/n}, \tag{29}$$

respectively.

3.2 Geometric infinite divisibility

Motivated by the stability property of the exponential distribution with respect to geometric compounding, Jammalamadaka and Kozubowski [6] introduced a notion of geometric infinite divisibility for angular distributions.

Definition 3.1 *An angular r.v.* Θ *is said to be geometric infinitely divisible if for any* $q \in (0,1)$ *there exist i.i.d. angular r.v.'s* $\Theta_1, \Theta_2, \ldots$ *such that*

$$\Theta_1 + \cdots + \Theta_{\nu_q} (mod\ 2\pi) \stackrel{d}{=} \Theta, \tag{30}$$

where ν_q *has the geometric distribution*

$$P(\nu_q = k) = (1-q)^{k-1} q, \quad k = 1, 2, 3 \ldots. \tag{31}$$

The following properties of wrapped exponential and Laplace distributions, established in [6, 7], follow from analogous results of classical exponential and Laplace laws.

Proposition 3.3 *If* $\Theta \sim WE(\lambda)$, *where* $\lambda \in R$, *then* Θ *is geometric infinitely divisible. Moreover, for any* $q \in (0,1)$ *the equality in distribution (30) holds where the* Θ_i*'s have the uniform circular distribution for* $\lambda = 0$ *and the* $WE(\lambda/q)$ *distribution for* $\lambda \neq 0$.

Proposition 3.4 *If* $\Theta \sim WL(\lambda, \kappa)$, *where* $\lambda, \kappa \geq 0$, *then* Θ *is geometric infinitely divisible. Moreover, for any* $q \in (0,1)$ *the equality in distribution (30) holds where the* Θ_i*'s have the uniform circular distribution for* $\lambda = 0$ *or* $\kappa = 0$, *and the* $WL(\lambda_q, \kappa_q)$ *distribution for* $\lambda, \kappa > 0$, *where*

$$\lambda_q = \lambda/\sqrt{q} \tag{32}$$

and κ_q *is the unique solution of the equation*

$$\frac{1}{\kappa_q} - \kappa_q = \sqrt{q}\left(\frac{1}{\kappa} - \kappa\right). \tag{33}$$

3.3 Maximum entropy property

Recall that the *entropy* of a r.v. Θ with p.d.f. f is defined as

$$H(\Theta) = -\int_0^{2\pi} f(\theta) \ln f(\theta) d\theta, \tag{34}$$

and provides a measure of uncertainty. Jaynes [9] proposed general inference procedures of finding a distribution that maximizes the entropy, and this method has been applied in a variety of fields including statistical mechanics, stock-market analysis, queuing theory, and reliability (see, e.g., [10]). It is well-known that if the mean direction and circular variance are fixed, then the entropy is maximized by the von Mises distribution, and under no restrictions on f the entropy is maximal for the circular uniform distribution (see, e.g., [10]). As shown in [6], the wrapped exponential distribution $WE(\lambda)$ maximizes the entropy under the condition

$$\int_0^{2\pi} \theta f(\theta) d\theta = m, \;\; 0 < m < 2\pi. \tag{35}$$

Proposition 3.5 *Consider the class C of all circular r.v.'s with density f satisfying the condition (35). Then, the maximum entropy is attained by the $WE(\lambda)$ distribution, where $\lambda = (2\pi\xi)^{-1}$ and ξ satisfies the equation*

$$\frac{m}{2\pi} = \xi - \frac{1}{e^{1/\xi} - 1}. \tag{36}$$

Moreover, the maximal entropy is

$$\max_{\Theta \in \mathcal{C}} H(\Theta) = \ln\left(\frac{1 - e^{-2\pi\lambda}}{\lambda}\right) + \frac{1}{\lambda} - \frac{2\pi e^{-2\pi\lambda}}{1 - e^{-2\pi\lambda}}. \tag{37}$$

Remark. By the monotonicity properties of the function $g(\xi) = \xi - (e^{1/\xi} - 1)^{-1}$, it follows that the distribution maximizing the entropy under the restriction (35) is wrapped exponential ($\lambda > 0$) for $0 < m < \pi$, wrapped negative exponential ($\lambda < 0$) for $\pi < m < 2\pi$, and circular uniform ($\lambda = 0$) for $m = \pi$.

Remark. This above result is analogous to the well-known property of the exponential distribution (which maximizes the entropy among all continuous probability distributions on $(0, \infty)$ with a given mean, see, e.g., [10]).

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