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On Sobolev tests of uniformity on the circle with an extension to the sphere

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Circular and spherical data arise in many applications, especially in Biology, Earth Sciences and Astronomy. In dealing with such data one of the preliminary steps before any further inference, is to test if such data is isotropic i.e. uniformly distributed around the circle or the sphere. In view of its importance, there is a considerable literature on the topic. In the present work we provide new tests of uniformity on the circle based on original asymptotic results. Our tests are motivated by the shape of locally and asymptotically maximin tests of uniformity against generalized von Mises distributions. We show that they are uniformly consistent. Empirical power comparisons with several competing procedures are presented via simulations. The new tests detect particularly well multimodal alternatives such as mixtures of von Mises distributions. A practically-oriented combination of the new tests with already existing Sobolev tests is proposed. An extension to testing uniformity on the sphere, along with some simulations, is included. The procedures are illustrated on a real dataset.

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1. Introduction

Directional statistics is a branch of Statistics that deals with observations that are directions or more generally observations lying on non-linear manifolds. In several applications these observations lie on the surface of the unit circle $\mathbb{S}^1 := \{\mathbf{u} \in \mathbb{R}^2, \mathbf{u}'\mathbf{u} = 1\}$ or the unit sphere $\mathbb{S}^2 := \{\mathbf{u} \in \mathbb{R}^3, \mathbf{u}'\mathbf{u} = 1\}$; throughout $\mathbf{v}'$ denotes the transpose of a vector $\mathbf{v} \in \mathbb{R}^p$ (similarly $\mathbf{V}'$ denotes the transpose of a matrix $\mathbf{V}$). Many applications of circular or spherical statistics can be found in various scientific domains such as Astronomy (see, e.g. Cuesta-Albertos et al., 2009; Faÿ et al., 2013), Biology (see, e.g. Batschelet, 1981; Couzin et al., 2005; Giunchi and Baldaccini, 2004; Golden et al., 2017; Morellato et al., 2010; Putman et al., 2014; Thomas et al., 2017) but also Geology, Medicine, Ecology. To quote Landler et al. (2018), "Circular data are common in biological studies. The most fundamental question that can be asked of a sample of circular data is whether it suggests that the underlying population is uniformly distributed around the circle, or whether it is concentrated around at least one preferred direction (e.g. a migratory goal or activity phase)".

When dealing with circular data, testing for uniformity or isotropy is therefore an essential first step before doing any further inference. If isotropy is indicated for a given sample, no modeling or estimation is needed, and the resulting sequence of observations can be seen as having no preferred direction or as noise on $\mathbb{S}^1$. Thus the uniform distribution occupies a central role amongst all circular distributions and as a result there is considerable literature on testing uniformity on the circle. Observations on the one dimensional manifold $\mathbb{S}^1$ can obviously be seen as realizations of a random variable taking values on $[0, 2\pi)$ that will be referred to as a circular random variable in the sequel. Letting $\Theta_1, \ldots, \Theta_n$, be a sequence of independently and identically distributed (i.i.d.) circular random variables, the most classical test of uniformity on the circle is the Rayleigh (1919) test that rejects the null hypothesis at the nominal level α when

$$2n^{-1}\big(\sum_{i=1}^{n} \sin(\Theta_i)\big)^2 + 2n^{-1}\big(\sum_{i=1}^{n} \cos(\Theta_i)\big)^2 > c_\alpha, \tag{1}$$

where, for large samples, $c_\alpha \approx \chi^2_{2;1-\alpha}$ with $\chi^2_{p;\nu}$ standing for the ν quantile of the chi-square distribution with p degrees of freedom. The Rayleigh test is a very simple test which is powerful to detect unimodal alternatives. However it is not *uniformly consistent* in the sense that it is blind against some alternatives to uniformity.

Since then a lot of work has been devoted to the problem in particular to obtain uniformly consistent tests. The Kuiper (1960) and Watson (1961) tests are the circular versions of the classical Kolmogorov–Smirnov and Cramér–von Mises tests respectively. Tests based on arc-lengths or spacings have been studied in Rao (1976). Beran (1969) and Giné (1975) studied the very broad class of Sobolev tests; see Section 2 for details on Sobolev tests. For more information on these and other classical tests for uniformity as well as for circular statistical analyses in general, the reader is referred to Mardia and Jupp (2000), Jammalamadaka and SenGupta (2001) or Ley and Verdebout (2017). See also a recent review paper on this topic by García-Portugués and Verdebout (2019).

Testing uniformity on the circle or on the sphere is still a very much studied problem: we mention Jupp (2001) who provided modifications of the classical Rayleigh test, Feltz and Goldin (2001) who proposed partition-based tests on the circle, Figueiredo (2007) who provided a comparison of various tests, while data-driven Sobolev tests for uniformity are obtained in Bogdan et al. (2002) and Jupp (2008). Cuesta-Albertos et al. (2009) proposed projection-based tests, Pycke (2010) obtained new Sobolev tests on the circle, and Lacour and Pham Ngoc (2014) considered uniformity tests on the sphere for data perturbed by random rotations. Jammalamadaka and Terdik (2019) used spherical harmonics for various tests of symmetry including uniformity on $\mathbb{S}^2$, while Cutting et al. (2017) studied tests of uniformity on high-dimensional spheres.

In general Sobolev test statistics of uniformity on the circle are based on weighted or unweighted sums of a fixed number M of terms involving empirical trigonometric moments. Here we show that the tests based on such unweighted sums of M terms are locally and asymptotically optimal within a parametric model; see Section 3 for details. However in order to obtain Sobolev tests that are uniformly consistent, it is well know that M has to be arbitrarily large. Many papers including the recent paper by Pycke (2010) considered weighted infinite sums of squared empirical trigonometric moments letting M diverge to infinity; see Section 2 for details . Although Pycke (2010) used weights to obtain converging series, the chosen sequences of weights converge slowly to zero in order to provide uniformly consistent tests that detect multimodal alternatives well. Bogdan et al. (2002) and Jupp (2008) used a different approach to reach uniform consistency. Rather than letting M diverge to infinity they considered a data-driven selection of M. While for large-M tests it seems to be a common belief that it is necessary to use weighted sums in order to get asymptotically valid and uniformly consistent tests, we show the contrary in the present work. We assume here that $M := M_n$ is a sequence of positive integers that diverges to infinity together with the sample size n. Inspired by high-dimensional techniques we obtain an original asymptotic result for unweighted sums of squared empirical trigonometric moments in this double asymptotic framework. We also show that the resulting tests are uniformly consistent; see Section 4 for details. Moreover a simulation study (see Section 5) indicates that our tests dominate all the competitors under various important multimodal alternatives. In Section 6, a test that combines the data-driven test of Bogdan et al. (2002) with the new test is proposed for practical purposes while in Section 7 an extension of the main result to the sphere $\mathbb{S}^2$ is obtained. A real–data example follows in Section 8. The proofs of the various results are collected in the Appendix and in a supplementary material. The latter supplementary material also includes simulation results related to the proposed test on $\mathbb{S}^2$.

2. Characteristic function and Sobolev tests

Let Θ denote an arbitrary circular random variable with an absolutely continuous circular distribution function $F_\Theta(\vartheta) := \mathbb{P}(\Theta \leq \vartheta)$, $\Theta \in [0, 2\pi)$ with respect to an arbitrary origin. We wish to test the null hypothesis of uniformity $\mathcal{H}_0 : F_\Theta(\vartheta) = \frac{\vartheta}{2\pi}$, for all $\vartheta \in [0, 2\pi)$. As already mentioned in the Introduction, the present problem plays a central role in directional statistics. In our approach for testing uniformity we will make use of the characteristic function of Θ which is defined for any integer $m \geq 1$ by

$$\varphi_\Theta(m) := \mathbb{E}\left[\cos(m\Theta)\right] + i\mathbb{E}\left[\sin(m\Theta)\right] =: \alpha_F(m) + i\beta_F(m); \tag{2}$$

$\varphi_\Theta(m)$ is also often called the mth trigonometric moment of F_Θ in the circular setup; in the sequel we also refer to $(\alpha_F(m), \beta_F(m))$ as the mth trigonometric moments of F_Θ. It is well known that for circular distributions the characteristic function needs to be defined only for integer values of the argument (see e.g. p.26 of Jammalamadaka and SenGupta (2001) or Meintanis and Verdebout (2018)). Thus writing $(\alpha_0(m), \beta_0(m))$ for the mth trigonometric moments *under the null hypothesis* $\mathcal{H}_0$ of uniformity, we have that $\alpha_0(m) = \beta_0(m) = 0$ and consequently that $\alpha_0^2(m) + \beta_0^2(m) = 0$ for any integer $m \geq 1$. Letting

$$\alpha_n(m) = \frac{1}{n}\sum_{j=1}^{n} \cos(m\Theta_j) \quad \text{and} \quad \beta_n(m) = \frac{1}{n}\sum_{j=1}^{n} \sin(m\Theta_j) \tag{3}$$

stand for the natural estimators of the trigonometric moments, the tests we study throughout the paper reject the null hypothesis $\mathcal{H}_0$ for large values of test statistics of the form

$$\begin{aligned} S_{n,w} &= 2n \sum_{m=1}^{\infty} \left(\alpha_n^2(m) + \beta_n^2(m)\right) w(m) \\ &= \frac{2}{n} \sum_{j,k=1}^{n} \sum_{m=1}^{\infty} \cos(m(\Theta_j - \Theta_k)) w(m), \end{aligned} \tag{4}$$

where $w(m)$ is a sequence of non-negative weights. The test statistics in (4) are weighted combinations of V–statistics that provide an important class of tests called the Sobolev tests (see e.g. García-Portugués and Verdebout, 2019). In this particular context of testing isotropy, Sobolev tests can be traced back to Beran (1969) and Rao (1972). Within this class one finds the Watson test that is obtained by taking $w(m) = m^{-2}$, $m \geq 1$, which shows power against any non-uniform alternative. Other tests belonging to the Sobolev class include the tests due to Rothman (1972), Bingham (1974), Giné (1975) and Hermans and Rasson (1985) to cite only a few. Provided that $\sum_{m=1}^{\infty} w(m) < \infty$, classical results on V-statistics imply that $S_{n,w}$ is asymptotically distributed as $\sum_{m=1}^{\infty} w(m) Y_m$, where $Y_1, Y_2, \ldots,$ is a sequence of independent chi-square random variables with two degrees of freedom.

Now the weights $w(1), w(2), \ldots$ in (4) can be given an intuitive interpretation. For instance if one chooses $w(1) = 1$ and $w(m) = 0$ for all $m \geq 1$, the resulting test is the classical Rayleigh test in (1) which is powerful against unimodal alternatives; it is indeed the uniformly most powerful invariant test against von Mises alternatives and the locally most powerful invariant test against symmetric wrapped stable families (see Jammalamadaka and SenGupta, 2001, Section 2.2.8), as well as the locally and asymptotically maximin test within a class of rotationally symmetric deviations (see Cutting et al., 2017). For the definition of a maximin test, see the comment just above Proposition 3.1. However these alternative distributions are all unimodal. In general the coefficient $w(m)$ can be interpreted as the weight associated with the eigenfunctions $\cos(m\vartheta)$ and $\sin(m\vartheta)$. Clearly the bigger the weight $w(m)$, the better the resulting Sobolev test in detecting differences involving the mth trigonometric moment. In particular the slower the sequence $w(m)$ converges to zero the better the resulting test will detect multimodal alternatives. This is the motivation underpinning the two Sobolev tests obtained by putting $w(m) = m^{-1}$ and $w(m) = a^{m-1}$ $(a < 1)$, respectively that have been suggested by Pycke (2010). The tests in Pycke (2010) are shown to perform well against multimodal alternatives. As suggested there, natural choices for the weights $w(1), w(2), \ldots$ are weights related to probability distributions on the set of positive natural numbers $\mathbb{N}_0 := \{1, 2, ...\}$. In these cases the series $C_w(\vartheta) = \sum_{m=1}^{\infty} \cos(m\vartheta) w(m)$, yields the value of the real part of the characteristic function corresponding to $w(\cdot)$ computed at the argument ϑ: for example the geometric distribution/weight $w_a(m) = (1-a)a^m, \ m \geq 1$, leads to

$$C_a(\vartheta) = (1-a)\left(\frac{1 - a\cos\vartheta}{1 - 2a\cos\vartheta + a^2}\right), \tag{5}$$

the positive Poisson distribution, the weight $w_a(m) = (1-e^{-a})^{-1} e^{-a} (a^m/m!), \ m \geq 1$, leads to $C_a(\vartheta) = (e^a - 1)^{-1} e^{a\cos\vartheta} \cos(a \sin\vartheta) - 1$, or the logarithmic distribution while $w_a(m) = c(a^m/m), \ m \geq 1$, with $c = -(\log(1-a))^{-1}$, leads to $C_a(\vartheta) = c\log(1 - 2a\cos\vartheta + a^2)^{-1/2}$.

3. Exponential model and optimal tests

In the present section we consider an absolutely continuous exponential family with densities of the form

$$\vartheta \to c_{\boldsymbol{\kappa}_M} \exp(\boldsymbol{\kappa}'_M \boldsymbol{\beta}(\vartheta)), \tag{6}$$

where $\boldsymbol{\beta}(\vartheta) := (\cos(\vartheta), \sin(\vartheta) \ldots, \cos(M\vartheta), \sin(M\vartheta))$, $\boldsymbol{\kappa}_M := (\kappa_1, \ldots, \kappa_{2M}) \in \mathbb{R}^{2M}$ is a vector of real parameters and $c_{\boldsymbol{\kappa}_M}^{-1} := \int_0^{2\pi} \exp(\boldsymbol{\kappa}_M' \boldsymbol{\beta}(\vartheta)) d\vartheta$ is a normalizing constant. This exponential model that can be traced back to Maksimov (1967) has been studied in Beran (1979) and Gatto and Jammalamadaka (2007). In the sequel this model is referred to as the generalized von Mises model; $\kappa_1 > 0$ and $\kappa_2 = \ldots = \kappa_{2M} = 0$ yields the very classical von Mises distribution. In the rest of the section we denote by $\mathrm{P}^{(n)}_{\boldsymbol{\kappa}_M;M}$ the joint distribution of $\Theta_1, \ldots, \Theta_n$, under density (6). In the exponential model (6) the score test of uniformity on $\mathbb{S}^1$ is the score test for $\mathcal{H}_0 : \boldsymbol{\kappa}_M = \mathbf{0}$ against $\mathcal{H}_1 : \boldsymbol{\kappa}_M \neq \mathbf{0}$ that is obviously based on the first M empirical trigonometric moments $(\alpha_n(1), \beta_n(1), \ldots, \alpha_n(M), \beta_n(M))$. More precisely the score test $\phi_M^{(n)}$ rejects the null hypothesis at the asymptotic nominal level α when

$$S_{1,M} := 2n \sum_{m=1}^{M} \left(\alpha_n^2(m) + \beta_n^2(m)\right) > \chi^2_{2M;1-\alpha}. \tag{7}$$

Note that $S_{1,M}$ is an unweighted and truncated (at M) version of $S_{n,w}$ in (4). The asymptotic distribution of $S_{1,M}$ under the null hypothesis can easily be obtained since a simple application of the Central Limit Theorem allows us to show that under the null hypothesis each term $2n\left(\alpha_n^2(m) + \beta_n^2(m)\right)$ is in the limit distributed as a chi-square random variable with two degrees of freedom and that any pair $2n\left(\alpha_n^2(m_i) + \beta_n^2(m_i)\right)$ and $2n\left(\alpha_n^2(m_j) + \beta_n^2(m_j)\right)$, $i \neq j$, is asymptotically independently distributed. In the next result, we complement the existing results by showing that the test $\phi_M^{(n)}$ is also locally and asymptotically maximin for testing $\mathcal{H}_0 : \boldsymbol{\kappa}_M = \mathbf{0}$ against $\mathcal{H}_1 : \boldsymbol{\kappa}_M \neq \mathbf{0}$ and by computing the limiting distribution of $S_{1,M}$ under local alternatives. In this connection recall that a given test ϕ^* is called maximin in the class $\mathcal{C}_\alpha$ of level–α tests for $\mathcal{H}_0$ against $\mathcal{H}_1$ if (i) ϕ^* has level α and (ii) the power of ϕ^* is such that

$$\inf_{\mathrm{P} \in \mathcal{H}_1} \mathbb{E}_{\mathrm{P}}[\phi^*] \geq \sup_{\phi \in \mathcal{C}_\alpha} \inf_{\mathrm{P} \in \mathcal{H}_1} \mathbb{E}_{\mathrm{P}}[\phi].$$

In the following result we show that the test $\phi_M^{(n)}$ is locally and asymptotically maximin against local alternatives $\mathrm{P}^{(n)}_{n^{-1/2}\mathbf{k}_M^{(n)};M}$, where $\mathbf{k}_M^{(n)}$ is a bounded sequence of $\mathbb{R}^{2M}$; for a precise definition of locally and asymptotically maximin tests, see, e.g. Chapter 5 of Ley and Verdebout (2017).

Proposition 3.1. *Fix $M \in \mathbb{N}_0$. For testing $\mathcal{H}_0 : \boldsymbol{\kappa}_M = \mathbf{0}$ against $\mathcal{H}_1 : \boldsymbol{\kappa}_M \neq \mathbf{0}$,*

(i) the test $\phi_M^{(n)}$ is locally and asymptotically maximin and

(ii) under local alternatives $\mathrm{P}^{(n)}_{n^{-1/2}\mathbf{k}_M^{(n)};M}$, where $\mathbf{k}_M^{(n)}$ is a bounded sequence of $\mathbb{R}^{2M}$ such that $\mathbf{k}_M := \lim_{n\to\infty} \mathbf{k}_M^{(n)}$, $S_{1,M}$ converges weakly to a chi-square random variable with $2M$ degrees of freedom and non-centrality parameter $\|\mathbf{k}_M\|^2/2$.

In Proposition 3.1, we show that $\phi_M^{(n)}$ is the locally and asymptotically maximin test for $\mathcal{H}_0 : \boldsymbol{\kappa}_M = \mathbf{0}$ against $\mathcal{H}_1 : \boldsymbol{\kappa}_M \neq \mathbf{0}$. The following result enables the computation of the local asymptotic power of the test $\phi_M^{(n)}$ under various local alternatives within the Beran (1979) family.

Proposition 3.2. *Fix $(M, M') \in \mathbb{N}_0^2$. Then*

(i) under $\mathrm{P}^{(n)}_{n^{-1/2}\mathbf{k}^{(n)}_{M'};M'}$ with $M' > M$ and $\mathbf{k}^{(n)}_{M'} := (\mathbf{k}^{(n)}_M, k^{(n)}_{2M+1}, \ldots, k^{(n)}_{2M'})'$ a bounded sequence of $\mathbb{R}^{2M'}$, $S_{1,M}$ is asymptotically chi-square with $2M$ degrees of freedom and non-centrality parameter $\|\mathbf{k}_M\|^2/2$, where $\mathbf{k}_M := \lim_{n\to\infty} \mathbf{k}^{(n)}_M$.

(ii) under $\mathrm{P}^{(n)}_{n^{-1/2}\mathbf{k}^{(n)}_{M'};M'}$ with $M' < M$ and $\mathbf{k}^{(n)}_{M'} := (k_1, \ldots, k_{2M'})'$ a bounded sequence of $\mathbb{R}^{2M'}$, $S_{1,M}$ is asymptotically chi-square with $2M$ degrees of freedom and non-centrality parameter $\|\mathbf{k}_{M'}\|^2/2$, where $\mathbf{k}_{M'} := \lim_{n\to\infty} \mathbf{k}^{(n)}_{M'}$.

Note in particular that point (i) of Proposition 3.2 entails that if $\mathbf{k}_M = \mathbf{0}$ and if $\lim_{n\to\infty}(k^{(n)}_{2M+1}, \ldots, k^{(n)}_{2M'}) \neq \mathbf{0}$, the limiting distribution of $S_{1,M}$ under the local alternatives in point (i) is a central chi-square with $2M$ degrees of freedom so that the resulting test is blind to the corresponding alternatives while from Proposition 3.1 the optimal test $S_{1,M'}$ has a limiting power in the interval $(\alpha, 1)$.

To obtain *uniformly consistent* tests, that is tests that are consistent against any fixed alternative, two possibilities naturally arise: the first one consists in providing a suitable data-driven selection of M . As explained in the Introduction this first solution has been studied in Bogdan et al. (2002) who provided a smooth test in which $w(m) = 1,\ m \leq \widehat{M}$ and $w(m) = 0,\ m > \widehat{M}$, where the threshold $\widehat{M}$ is data-driven; see also its multivariate extension by Jupp (2008). The empirical selection procedure proposed by Bogdan et al. (2002) is based on results in Ledwina (1994). In the next section we investigate a second solution. More precisely we provide a new test based on the investigation under the null hypothesis of the asymptotic behavior of a standardized version of the unweighted sum

$$S_{1,M_n} := 2n \sum_{m=1}^{M_n} \left(\alpha_n^2(m) + \beta_n^2(m)\right), \tag{8}$$

when $M_n \to \infty$, as $n \to \infty$.

4. A new Sobolev test

In the present section, we provide a new test $\phi^{(n)}_{\mathrm{MRV}}$ of uniformity on the circle. As explained at the end of the previous section the test $\phi^{(n)}_{\mathrm{MRV}}$ will be based on a standardized version of the quantity S_{1,M_n} in (8). More precisely we consider as test statistic the standardized version

$$S_{1,M_n}^{\text{stand}} \quad := \quad \frac{S_{1,M_n} - 2M_n}{2\sqrt{M_n}}, \tag{9}$$

of S_{1,M_n} for which the following result provides the asymptotic null distribution as $M_n \to \infty$ with $M_n = o(n^2)$, as $n \to \infty$.

Proposition 4.1. *Let M_n be a sequence of integers such that $M_n \to \infty$ with $M_n = o(n^2)$ as $n \to \infty$. Then, under $\mathcal{H}_0$, S_{1,M_n}^{stand} converges weakly to standard normal random variable as $n \to \infty$.*

Proposition 4.1 shows that once $S_{1,M}$ in (8) is properly rescaled, it is possible to obtain its asymptotic behavior with a large number M_n of weights $w(m)$ that are equal to one provided that M_n is not too large with respect to n. The resulting test $\phi_{\text{MRV}}^{(n)}$ rejects the null hypothesis at the asymptotic level α when $S_{1,M_n}^{\text{stand}} > z_{1-\alpha}$ where z_ν stand for the ν-quantile of the standard Gaussian distribution.

In order to illustrate numerically Proposition 4.1, we generated 5000 samples of uniform random vectors on $\mathbb{S}^1$ with various sample sizes $n = 30, 100, 500, 1000$. For each replication, we computed the test statistic $S_{1,M}^{\text{stand}}$ with various values of $M = 30, 100, 500, 1000$. In Figure 1 we provide standard Gaussian qq-plots of the resulting 5000 values of $S_{1,M}^{\text{stand}}$ for any pair (n, M). In Figure 2 we provide the empirical type-I risks (i.e., rejection rates under the null hypothesis) of the tests $\phi_{\text{MRV}}^{(n)}$ performed at the nominal level .05 for any pair (n, M); more precisely letting $(S_{1,M}^{\text{stand}})_k$ stand for the value of $S_{1,M}^{\text{stand}}$ obtained from the kth replication, $k = 1, \ldots, K = 5000$, we computed

$$r_{n,M}^{(.05)} := \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}[(S_{1,M}^{\text{stand}})_k > z_{.95}],$$

where $\mathbb{I}[.]$ denotes the classical indicator function. Obviously, a value close to .05 for $r_{n,M}^{(.05)}$ indicates a correct type-I risk (the expectation of the test under the null hypothesis) for $\phi_{\text{MRV}}^{(n)}$ performed at the nominal level .05. While inspection of Figure 1 clearly reveals that the Gaussian approximation is better when the sample size is greater than M, Figure 2 shows that in practice, the type-I risk of $\phi_{\text{MRV}}^{(n)}$ is not drastically influenced by the choice of M. Indeed when $M = 1000$ and $n = 30$, the resulting empirical type-I risk of $\phi_{\text{MRV}}^{(n)}$ is .068. Hence choosing M not too large with respect to n seems to be a good rule of thumb in order to get an asymptotically valid test.

In the next result we show the uniform consistency of the test within a very natural class of alternatives, also considered in Pycke (2010).

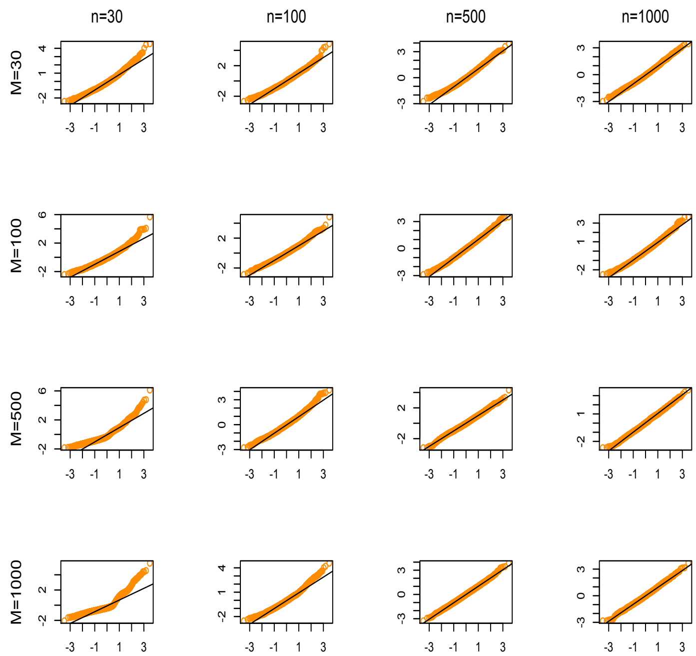

Figure 1. Standard Gaussian qq–plots of $S_{1,M}^{\mathrm{stand}}$ for various values of n and M based on 5000 replications.

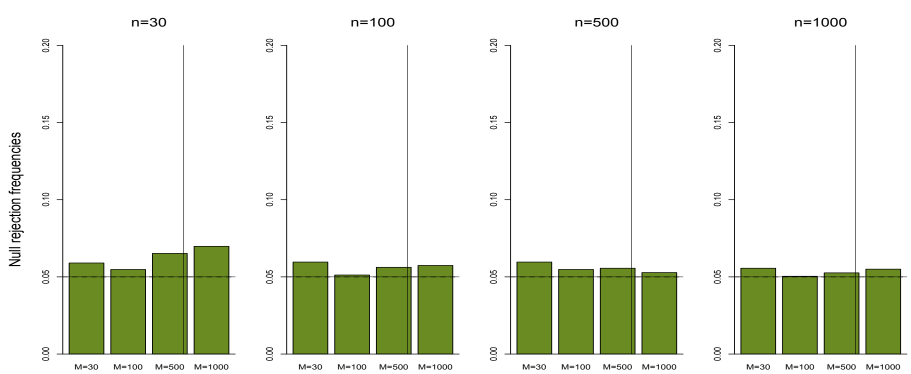

Figure 2. Estimated type-I risks (null rejection frequencies) of the $\phi_{\mathrm{MRV}}^{(n)}$ test performed at the nominal level $\alpha = .05$ for various values of n and M based on 5000 replications.

Proposition 4.2. *Let $\mathcal{F}$ denote a class of distributions with trigonometric moments $\varphi_\Theta(m) = \alpha_F(m) + i\beta_F(m)$, such that $0 < \sum_{m=1}^{\infty} \big(\alpha_F^2(m) + \beta_F^2(m)\big) < +\infty$ for all $F \in \mathcal{F}$. Then the test $\phi^{(n)}_{\rm MRV}$ is uniformly consistent within the class $\mathcal{F}$.*

In the next section we compare the empirical power of the test $\phi^{(n)}_{\rm MRV}$ with that of several competitors.

5. Power comparisons

The objective of the present section is to compare the empirical power of the proposed test with several competitors. Before starting the description of the simulation schemes we recall that a von Mises distribution with a density of the form $\vartheta \to c_\kappa \exp(\kappa \cos(\vartheta - \mu))$, where $\mu \in [0, 2\pi)$ is a location parameter, $\kappa > 0$ is a concentration parameter and c_κ is a normalizing constant (as mentioned already note that the latter density with $\mu = 0$ can be obtained by setting $\boldsymbol{\beta}(\vartheta) = \cos(\vartheta)$ in (6)). To perform our empirical comparison, we generated $N = 1,500$ samples of i.i.d. circular random variables

$$\Theta^{(\rho)}_{\ell;j}, \quad \rho = 1, \ldots, 8,\ \ell = 0, \ldots, 3,\ j = 1, \ldots, n = 100,$$

where the $\Theta^{(1)}_{\ell;j}$'s have a von Mises distribution with location 0 and concentration parameter $\ell/6$, the $\Theta^{(\rho)}_{\ell;j}$'s for $\rho = 2, \ldots, 4$, are distributed as mixtures of the form $\sum_{j=1}^{m} \epsilon_j \Theta_j$ where $\epsilon_j := \mathbb{I}_{(j-1)/m \leq U \leq j/m}$, with $U \sim {\rm unif}[0,1]$ and where Θ_j has a von Mises distribution with location $2\pi(j-1)/m$. The $\Theta^{(2)}_{\ell;j}$'s are obtained taking $m = 4$ and concentration 3ℓ, the $\Theta^{(3)}_{\ell;j}$'s are obtained taking $m = 8$ and concentration 15ℓ while the $\Theta^{(4)}_{\ell;j}$'s are obtained taking $m = 16$ and concentration 50ℓ. The $\Theta^{(\rho)}_{\ell;j}$'s for $\rho = 5, \ldots, 8$ are distributed as generalized von Mises random variables in (6); the $\Theta^{(5)}_{\ell;j}$'s are obtained taking $M = 4$ and $\boldsymbol{\kappa}_M = (\ell/9)\mathbf{1}_{2M}$ ($\mathbf{1}_k := (1, \ldots, 1) \in \mathbb{R}^k$), the $\Theta^{(6)}_{\ell;j}$'s are obtained taking $M = 8$ and $\boldsymbol{\kappa}_M = (\ell/10)\mathbf{1}_{2M}$, the $\Theta^{(7)}_{\ell;j}$'s are obtained taking $M = 12$ and $\boldsymbol{\kappa}_M = (\ell/12)\mathbf{1}_{2M}$ while the $\Theta^{(8)}_{\ell;j}$'s are obtained taking $M = 16$ and $\boldsymbol{\kappa}_M = (\ell/16)\mathbf{1}_{2M}$. The value $\ell = 0$ always yields a uniform distribution while the values $\ell = 1, 2, 3$, provide distributions that are increasingly away from uniformity.

The resulting rejection frequencies of the following tests, all performed at nominal level 5%, are plotted in Figures 3 and 4: the well-known Rayleigh test $\phi^{(n)}_{\rm Ray}$, the Pycke (2010) test $\phi^{(n)}_{\rm Pyc}$ based on h_4 (see equation (10) in Pycke, 2010), the Bogdan et al. (2002) test $\phi^{(n)}_{\rm Bog}$ performed using the practical remarks of Section 5 of Bogdan et al. (2002), the Rao spacings test $\phi^{(n)}_{\rm Rao}$ (see Rao, 1976), the test based on $S_{n,w}$ with Geometric distribution weights in (5) with $a = 3/4$ and the test $\phi^{(n)}_{\rm MRV}$ based on $S^{\rm stand}_{1,M_n}$ performed with $M_n = 30$.

The critical values used for the test based on $S_{n,w}$ were computed numerically using 5000 replications.

Inspection of Figures 3 and 4 reveals first that all the tests respect the .05 nominal level constraint and that they are therefore valid. As expected the Rayleigh test dominates all the other tests in the von Mises case. The test based on $S_{n,w}$ with geometric distribution weights enjoys a nice global behavior while for mixtures of $m \geq 8$ von Mises distributions and for generalized von Mises distributions with $M \geq 8$, the new test $\phi^{(n)}_{\mathrm{MRV}}$ dominates the competitors.

6. A new test for practical use

As shown in the simulation results of the previous section, the test $\phi^{(n)}_{\mathrm{MRV}}$ based on $S^{\mathrm{stand}}_{1,M_n}$ clearly behaves well when the alternative hypothesis is multimodal. On the other hand it is not powerful against unimodal alternatives. Following the discussion at the end of Section 2, this might have been expected. Another point that may be seen as an issue in practice is the selection of M_n since a priori the asymptotic normality of $S^{\mathrm{stand}}_{1,M_n}$ is guaranteed for any choice of M_n as long as $M_n = o(n^2)$. For instance $M_n = \lfloor n^\gamma \rfloor$ with $\gamma \in (0, 2)$ are possible choices. Clearly, the larger one selects M_n the more the test will detect alternatives with a lot of modes. Unfortunately, taking M_n very large will decrease the power of the test against unimodal alternatives. Therefore, in a sense, the selection of the sequence M_n that diverges to ∞ should be driven by the alternative which is arguably unrealistic in practice. It is nevertheless exactly the same nice idea underpinning the data-driven selection $\widehat{M}$ of M by Bogdan et al. (2002) and Jupp (2008). The fact that the test $\phi^{(n)}_{\mathrm{Bog}}$ is dominated by $\phi^{(n)}_{\mathrm{MRV}}$ under multimodal alternatives theoretically comes from the fact that as shown in Bogdan et al. (2002) and Jupp (2008) $\lim_{n\to\infty} \mathrm{P}[\widehat{M} < \infty] = 1$ so that $\widehat{M}$ will diverge to infinity with probability zero. Now in practice the algorithm proposed to compute the data-driven $\widehat{M}$ as explained in Bogdan et al. (2002) is as follows:

(i) Compute $L(M) := S_{1,M} - 2M \log n$ for $1 \leq M \leq M_{\mathrm{thr}}$, where (quoting Bogdan et al. (2002)), "in theory M_{thr} is ∞ but simulations suggest that in practice it is sufficient to take $M_{\mathrm{thr}} = 10$ for $n \leq 100$"

(ii) Compute $\widehat{M} = \mathrm{argmax}_{1 \leq M \leq M_{\mathrm{thr}}} L(M)$

In a sense, the question of the selection of the threshold M_{thr} can be compared to the question of selecting M_n in $S^{\mathrm{stand}}_{1,M_n}$ since in theory M_{thr} has to be taken as $+\infty$. Note that Jupp (2008) suggested to take $M_{\mathrm{thr}} = 5$ in practice. Discussing the selection of the sequence M_n that diverges to ∞ is therefore a delicate issue. Again, the larger M_n, the more multimodal alternatives the test will be able to detect but at the cost of giving up power against unimodal alternatives.

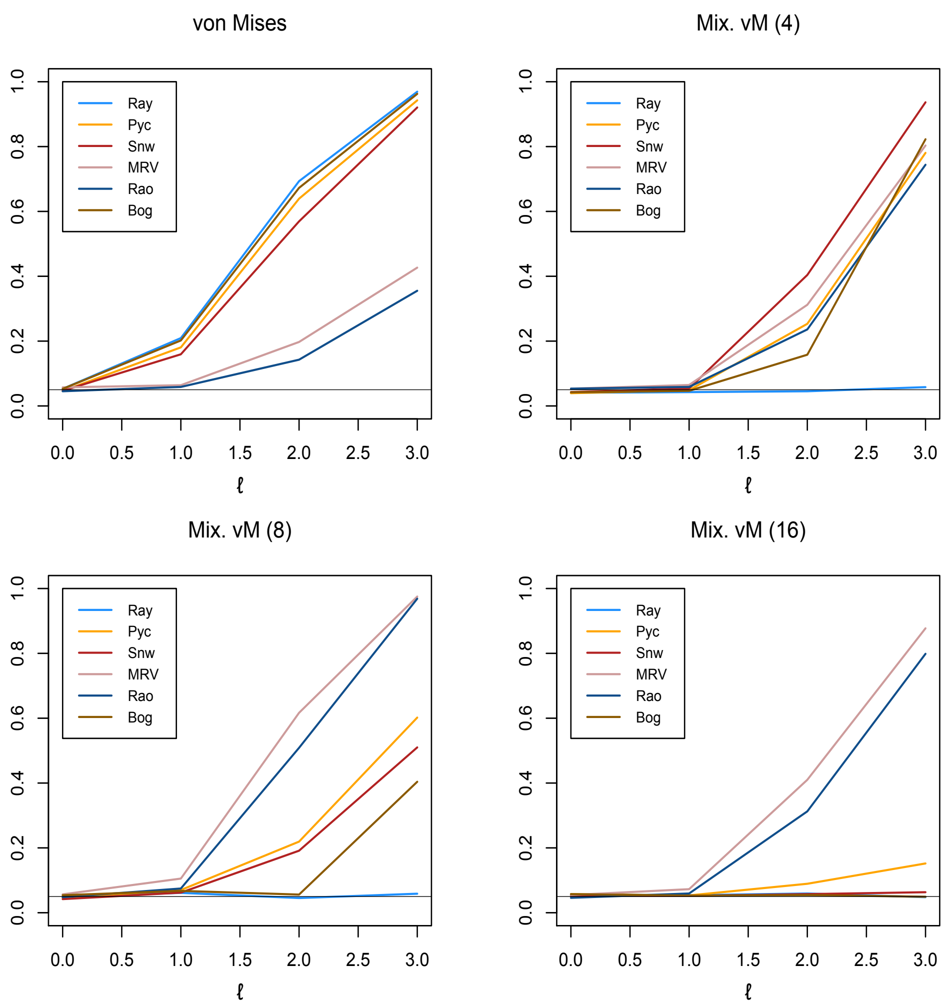

Figure 3. Empirical rejection probabilities of various tests of uniformity under various alternatives: (Top Left) von Mises, (Top Right) mixture of four von Mises distributions, (Bottom Left) mixture of eight von Mises distributions, (Bottom Right) mixture of sixteen von Mises distributions.

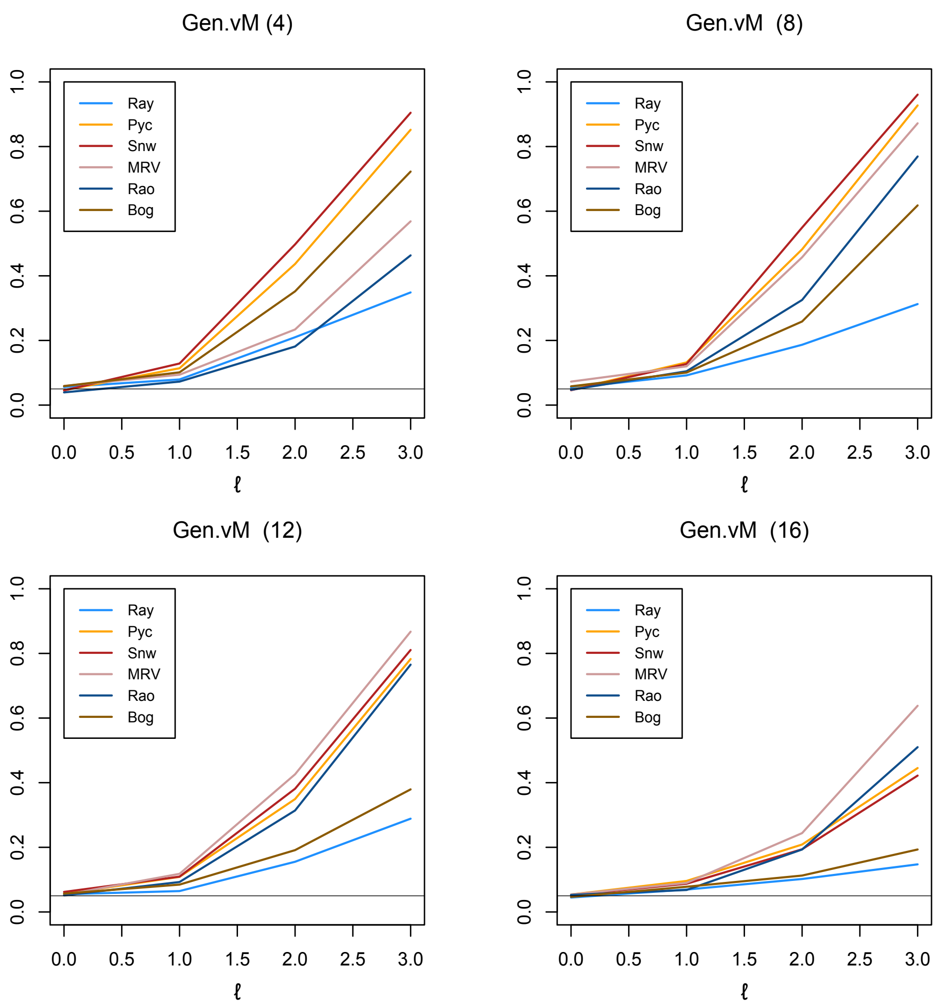

Figure 4. Empirical rejection probabilities of various tests of uniformity under various alternatives: (Top Left) generalized von Mises distribution with $M = 4$, (Top Right) generalized von Mises distribution with $M = 8$, (Bottom Left) generalized von Mises distribution with $M = 12$, (Bottom Right) generalized von Mises distribution with $M = 16$.

However, even with M_n being relatively small, the power of $\phi^{(n)}_{\rm MRV}$ against unimodal alternatives remains an issue. We propose here a new test that uses together the data-driven selection of M explained above as well as the test $\phi^{(n)}_{\rm MRV}$. The idea underpinning the new test finds its roots in the *power enhancement principle* studied in Fan et al. (2015) and Kock and Preinerstorfer (2019). The new test $\widetilde{\phi}^{(n)}$ is based on a combination of the two tests $\phi^{(n)}_{\rm Bog}$ and $\phi^{(n)}_{\rm MRV}$ given by

$$\widetilde{\phi}^{(n)} := \mathbb{I}_{[\widetilde{M}<V_{\rm thr}]}\phi^{(n)}_{\rm Bog} + \mathbb{I}_{[\widetilde{M}\geq V_{\rm thr}]}\phi^{(n)}_{\rm MRV}, \tag{10}$$

with the natural number $\widetilde{M}$ defined as $\widetilde{M} := \max_{1\leq j\leq n}\widehat{M}_j$, where $\widehat{M}_j \in \mathbb{N}_0$ is obtained by performing the algorithm for the selection of M described above based on the sample $\Theta_1,\ldots,\Theta_{j-1},\Theta_{j+1},\ldots,\Theta_n$; that is leaving Θ_j out of the sample. The threshold $V_{\rm thr}\in\mathbb{N}_0$ in (10) can be chosen by the practitioner. Any choice of $V_{\rm thr}$ yields a test with the correct asymptotic size. Note that if $V_{\rm thr}$ is taken larger than $M_{\rm thr}$ in the algorithm above, $\widetilde{\phi}^{(n)}$ is simply equivalent to $\phi^{(n)}_{\rm Bog}$ while if $V_{\rm thr}=1$, then $\widetilde{\phi}^{(n)}$ is equivalent to $\phi^{(n)}_{\rm MRV}$. The idea underpinning the test $\widetilde{\phi}^{(n)}$ is that if $\widetilde{M}$ is small (smaller than $V_{\rm thr}$), then $\widetilde{\phi}^{(n)}$ is equivalent to $\phi^{(n)}_{\rm Bog}$. On the other hand if $\widetilde{M}$ is larger than $V_{\rm thr}$, $\widetilde{\phi}^{(n)}$ is equivalent to $\phi^{(n)}_{\rm MRV}$. The practical motivation of $\widetilde{\phi}^{(n)}$ is therefore to improve the power of $\phi^{(n)}_{\rm MRV}$ under unimodal alternatives (or alternatives with a small number of modes) or equivalently to improve the power of $\phi^{(n)}_{\rm Bog}$ under multimodal alternatives, while keeping its nice properties under unimodal alternatives.

In order to illustrate the finite–sample properties of the test we performed the following simulation exercise: based on samples of size $n=50$, we performed the tests $\phi^{(n)}_{\rm Bog}$ at level .05 taking $M_{\rm thr}=30$ in the selection of M algorithm and the test $\phi^{(n)}_{\rm MRV}$ at level .05 taking $M_n=30$. Note that we took here $M_{\rm thr}$ as large as M_n to perform a fair comparison. We then performed the test $\widetilde{\phi}^{(n)}$ with $V_{\rm thr}=5$. We generated $N=1,500$ mutually independent samples of i.i.d. circular random variables

$$\Theta^{(\rho)}_{\ell;j}, \quad \rho=1,\ldots,4,\ \ell=0,\ldots,3,\ j=1,\ldots,n,$$

where the $\Theta^{(1)}_{\ell;j}$'s have a von Mises distribution with location 0 and concentration parameter $\ell/4$, while the $\Theta^{(\rho)}_{\ell;j}$'s for $\rho=2,3,4$, are distributed as a mixtures of the form $\sum_{j=1}^m \epsilon_j\Theta_j$ already defined in Section 5. Specifically the $\Theta^{(2)}_{\ell;j}$'s are obtained taking $m=4$ and concentration 6ℓ, the $\Theta^{(3)}_{\ell;j}$'s are obtained taking $m=8$ and concentration 25ℓ and the $\Theta^{(4)}_{\ell;j}$'s are obtained taking $m=12$ and concentration 200ℓ.

In Figure 5, we provide the empirical powers of the three tests: $\phi^{(n)}_{\rm Bog}$ (orange curve), $\phi^{(n)}_{\rm MRV}$ (darkgreen curve) and $\widetilde{\phi}^{(n)}$ (blue curve). Inspection of Figure 5 shows that when the number of modes under the alternative is low, the tests $\phi^{(n)}_{\rm Bog}$ and $\widetilde{\phi}^{(n)}$ are equivalent;

the blue curve and the orange curve cannot be distinguished in the top left and top right plots. In such situations, $\phi^{(n)}_{\text{Bog}}$ and $\widetilde{\phi}^{(n)}$ dominate $\phi^{(n)}_{\text{MRV}}$ as expected. However, when the number of modes is larger, $\phi^{(n)}_{\text{MRV}}$ dominates both $\phi^{(n)}_{\text{Bog}}$ and $\widetilde{\phi}^{(n)}$. Nevertheless, we can see that $\widetilde{\phi}^{(n)}$ performs better than $\phi^{(n)}_{\text{Bog}}$ in both bottom left and right plots. On the basis of this behavior of $\widetilde{\phi}^{(n)}$ and in view of the fact that there is no omnibus test that is uniformly most powerful against all possible alternatives (see Janssen (2000), and Escanciano (2009)), the combined test may seem a good compromise.

7. Extension to $\mathbb{S}^2$

In view of (4), the test statistic S_{1,M_n} in (8) may be interpreted geometrically in the Hilbert space $L^2(\mathbb{S}^1, \mu)$ of square-integrable real functions on $\mathbb{S}^1$ with respect to μ, the uniform probability measure on $\mathbb{S}^1$ equipped with the inner product

$$\langle f, g\rangle := \int_{\mathbb{S}^1} f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})$$

for $f, g \in L^2(\mathbb{S}^1, \mu)$. To this end note that a well–known orthonormal basis of $L^2(\mathbb{S}^1, \mu)$ is $\{\sqrt{2}e_m\}_{m\in\mathbb{Z}}$, where, in spherical coordinates, $e_m(\theta) := e^{i2\pi m\theta}$ or equivalently $e_m(\theta) = (\cos(m\theta), \sin(m\theta))$. With this in mind, let $\mathcal{E}_m$ be the two dimensional subspace of $L^2(\mathbb{S}^1, \mu)$ spanned by the orthonormal functions $g_{1,m} : \theta \to \sqrt{2}\cos(m\theta)$ and $g_{2,m} : \theta \to \sqrt{2}\sin(m\theta)$ and consider a mapping $t^{(1)}_m : [0, 2\pi] \to \mathcal{E}_m$ that maps a random angle Θ_i onto $t^{(1)}_m(\Theta_i) := g_{1,m}(\Theta_i)g_{1,m} + g_{2,m}(\Theta_i)g_{2,m}$. Then standard properties of inner products entail that S_{1,M_n} can be rewritten as

$$\begin{aligned} S_{1,M_n} &= \frac{1}{n}\sum_{i,j=1}^{n}\sum_{m=1}^{M_n}\langle t^{(1)}_m(\Theta_i), t^{(1)}_m(\Theta_j)\rangle \\ &= \frac{1}{n}\left\|\sum_{i=1}^{n} t^{(1)}_{(M_n)}(\Theta_i)\right\|^2_{L^2}, \end{aligned} \tag{11}$$

where the mapping $t^{(1)}_{(M_n)} : [0, 2\pi] \to L^2(\mathbb{S}^1, \mu)$ is defined as $t^{(1)}_{(M_n)}(\theta) := \sum_{m=1}^{M_n} t^{(1)}_m(\theta)$. Therefore the geometric interpretation provided here finds its roots in the fact that the space $L^2(\mathbb{S}^1, \mu)$ admits a Hilbert sum decomposition $L^2(\mathbb{S}^1, \mu) = \bigoplus_{m=0}^{\infty} \mathcal{E}_m$ into pairwise orthogonal subspaces $\mathcal{E}_m$ that contain orthonormal functions.

Following Jupp (2008), Sobolev test statistics on $\mathbb{S}^{p-1}$ with $p \geq 2$ (or more generally on Riemanian manifolds) can be obtained similarly by constructing an orthonormal basis of $L^2(\mathbb{S}^{p-1}, \mu)$, where μ now denotes here the uniform probability measure on $\mathbb{S}^{p-1}$. Such orthonormal bases can be obtained via the eigenfunctions associated with the non-zero

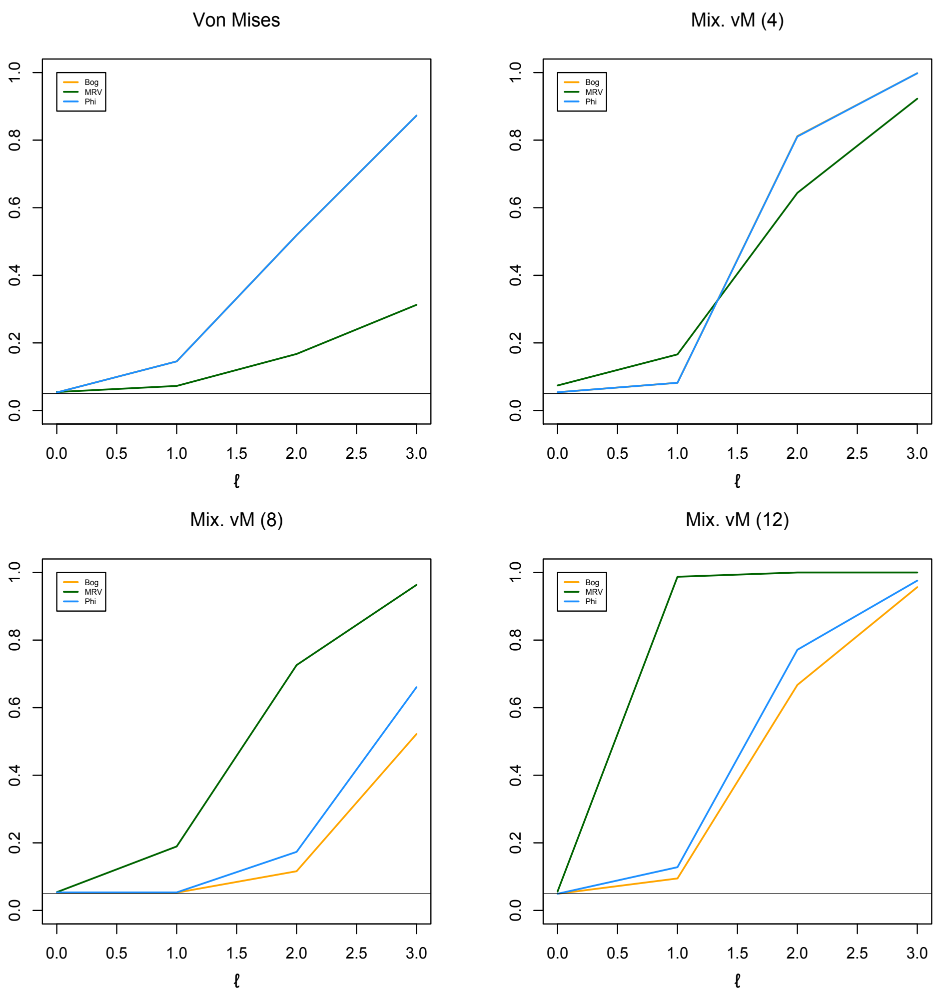

Figure 5. Empirical rejection probabilities of $\phi^{(n)}_{\mathrm{Bog}}$ (orange), $\phi^{(n)}_{\mathrm{MRV}}$ (darkgreen) and $\widetilde{\phi}^{(n)}$ (blue) under various alternatives: (Top Left) von Mises distributions, (Top Right) mixture of four von Mises distributions, (Bottom Left) mixture of eight von Mises distributions, (Bottom Right) mixture of twelve von Mises distributions.

eigenvalues of the *Laplace operator* Δ acting on $\mathbb{S}^{p-1}$. For more details on the Laplace operator on closed smooth Riemanian manifolds, see for instance Giné (1975) and the references therein. Mimicking the construction above on $\mathbb{S}^1$, denote by $\mathcal{E}_m$ the space of eigenfunctions $\mathbb{S}^{p-1} \to \mathbb{R}$ corresponding to the m-th non-zero eigenvalue of the Laplacian Δ, with dimension $d_{p,m} := \dim \mathcal{E}_m$. There exists a well-defined mapping $\mathrm{t}_m^{(p-1)} : \mathbb{S}^{p-1} \to \mathcal{E}_m$ that can be written as $\mathrm{t}_m^{(p-1)}(\mathbf{u}) := \sum_{i=1}^{d_{p,m}} g_{i,m}(\mathbf{u}) g_{i,m}$, where $\{g_{i,m}\}_{i=1}^{d_{p,m}}$ form an orthonormal basis of $\mathcal{E}_m$. The function $\mathbf{u} \mapsto \mathrm{t}_{(M_n)}^{(p-1)}(\mathbf{u}) := \sum_{m=1}^{M_n} \mathrm{t}_m^{(p-1)}(\mathbf{u})$ is a mapping from $\mathbb{S}^{p-1}$ to the Hilbert space $L^2(\mathbb{S}^{p-1}, \mu)$. Based on n observations $\mathbf{U}_1, \ldots, \mathbf{U}_n$ on $\mathbb{S}^{p-1}$, a Sobolev test rejects $\mathcal{H}_0$ for large values of the test statistic

$$\begin{aligned} S_{p-1,M_n} &:= \frac{1}{n} \left\| \sum_{i=1}^{n} \mathrm{t}_{(M_n)}^{(p-1)}(\mathbf{U}_i) \right\|_{L^2}^2 = \frac{1}{n} \sum_{i,j=1}^{n} \langle \mathrm{t}_{(M_n)}^{(p-1)}(\mathbf{U}_i), \mathrm{t}_{(M_n)}^{(p-1)}(\mathbf{U}_j) \rangle \\ &= \frac{1}{n} \sum_{i,j=1}^{n} \sum_{m=1}^{M_n} \langle \mathrm{t}_m^{(p-1)}(\mathbf{U}_i), \mathrm{t}_m^{(p-1)}(\mathbf{U}_j) \rangle, \end{aligned} \tag{12}$$

where $\langle f, g \rangle$ denotes the inner product on $L^2(\mathbb{S}^{p-1}, \mu)$. Note that the second equality in (12) follows from $\langle f, g \rangle = 0$ for any $f \in \mathcal{E}_k, g \in \mathcal{E}_l$, $k \neq l$, due to the definition of the $\mathcal{E}_k$'s. Clearly from the discussion above we have that the circular test statistic S_{1,M_n} corresponds to S_{p-1,M_n} with $p = 2$.

In the spherical case (on $\mathbb{S}^2$) an explicit expression for S_{2,M_n} can be obtained using the fact that for $\mathbf{u}, \mathbf{v} \in \mathbb{S}^2$,

$$\langle \mathrm{t}_m^{(2)}(\mathbf{u}), \mathrm{t}_m^{(2)}(\mathbf{v}) \rangle = (2m+1) P_m(\mathbf{u}'\mathbf{v}),$$

where P_m denotes the Legendre polynomial of order m. For $p \geq 4$, similar expressions can be obtained and involve Gegenbauer polynomials. The following result is the analog of Proposition 4.1 in the spherical case.

Proposition 7.1. *Let M_n be a sequence of integers such that $M_n \to \infty$ with $M_n = o(n^{2/3})$ as $n \to \infty$. Then, under the null hypothesis of uniformity on $\mathbb{S}^2$,*

$$S_{2,M_n}^{\text{stand}} := \frac{S_{2,M_n} - M_n(M_n+2)}{\sqrt{2M_n(M_n+2)}}$$

converges weakly to a standard normal random variable as $n \to \infty$.

Proposition 7.1 provides a natural extension of the test $\phi_{\text{MRV}}^{(n)}$ to $\mathbb{S}^2$. Note that the assumption $M_n = o(n^2)$, $n \to \infty$, in Proposition 4.1 that allows us to obtain the asymptotic normality of the standardized test statistic on $\mathbb{S}^1$ is less restrictive than the standing assumption $M_n = o(n^{2/3})$, which guarantees the asymptotic normality of its $\mathbb{S}^2$ counterpart. A generalization of Proposition 7.1 to tests on $\mathbb{S}^{p-1}$ with $p \geq 4$ remains unclear.

However most practical applications on this topic lie on $\mathbb{S}^1$ and $\mathbb{S}^2$ which are covered by Propositions 4.1 and 7.1. Finally note that a practical test similar to the one provided in Section 6 can readily be obtained by combining the test that rejects at the (asymptotic) level α when $S^{\text{stand}}_{2,M_n} > z_{1-\alpha}$ with the test based on the data–driven selection of M provided in Jupp (2008).

8. Real data illustration

As mentioned in the Introduction, circular data are common in biological studies. In the present section, we illustrate the proposed methods on a real data set obtained in Giunchi and Baldaccini (2004) who studied migratory orientation of hirundines. More precisely, the latter paper reports on a study of the orientation of the barn swallow (Hirundo rustica), a typical diurnal trans-Saharan migrant. The major aim of their study is to examine the role of visual and magnetic cues in juvenile swallows during their first migratory journey. In this illustration we consider the orientations of a control group of swallows under overcast conditions. The data plotted in Figure 6 consists of $n = 66$ orientations.

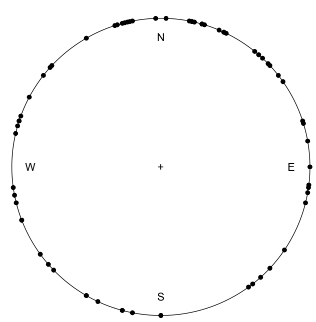

Figure 6. Orientations of a control group of $n = 66$ swallows.

Although inspection of Figure 6 shows a very weak tendency for hirundines to choose the north direction, non-uniformity seems however difficult to detect. We performed the following analysis on the dataset: we considered the n samples of 65 orientations obtained by omitting each time one observation from the sample. On each of these n samples we performed various tests of uniformity: the Rayleigh test, the test $\phi^{(n)}_{\text{Bog}}$ of Bogdan et al.

(2002) taking $M_{\text{thr}} = 30$ in the selection of M algorithm described in Section 6, the test $\phi^{(n)}_{\text{MRV}}$ with $M_n = 30$ and the test $\tilde{\phi}^{(n)}$ with $V_{\text{thr}} = 5$ (see Section 6 for details). In Figure 7, we provide boxplots of the asymptotic p-values of the four tests. Inspection of Figure 7 clearly reveals that the p-values of the Rayleigh test, $\phi^{(n)}_{\text{Bog}}$ and $\tilde{\phi}^{(n)}$ are the same. This comes from the fact that for all samples the selection of M algorithm provides $\widehat{M} = 1$ so that $\phi^{(n)}_{\text{Bog}}$ is equivalent to the Rayleigh test. For all samples we also obtain that $\tilde{\phi}^{(n)}$ is equivalent to $\phi^{(n)}_{\text{Bog}}$. The three tests do not reject the null hypothesis of uniformity. In contrast, the test $\phi^{(n)}_{\text{MRV}}$ strongly rejects the null hypothesis of uniformity. As a conclusion when using a model to describe the orientations of this group of hirundines, one should definitely select a multimodal model.

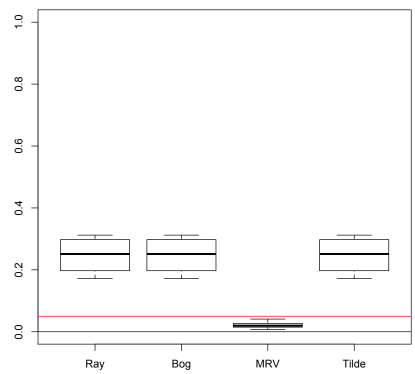

Figure 7. Boxplots of the 66 p-values of the Rayleigh test, $\phi^{(n)}_{\text{Bog}}$, $\phi^{(n)}_{\text{MRV}}$ and $\tilde{\phi}^{(n)}$.

Supplementary Material

Supplement to "On Sobolev tests of uniformity on the unit circle with an extension to the unit sphere"
(doi: completed by the typesetter; .pdf). In the supplementary material, we provide some further simulations related to the proposed test on $\mathbb{S}^2$ and the proofs of Propositions 3.1, 3.2 and 4.2.

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Appendix A: Proofs of Propositions 4.1 and 7.1

The appendix collects the proofs of Propositions 4.1 and 7.1. As mentioned above, the proofs of Propositions 3.1, 3.2 and 4.2 are provided in the supplementary material. As explained in Section 7, the Sobolev statistics we investigate in the paper may be rewritten as

$$\begin{aligned} S_{1,M_n} &= \frac{1}{n}\sum_{i,j=1}^{n}\langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_j)\rangle \\ &= \frac{2}{n}\sum_{1\le i<j\le n}\langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_j)\rangle + \frac{1}{n}\sum_{i=1}^{n}\langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_i)\rangle \\ &= \frac{2}{n}\sum_{1\le i<j\le n}\langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_j)\rangle + 2M_n \end{aligned} \tag{13}$$

and

$$\begin{aligned} S_{2,M_n} &:= \frac{1}{n}\sum_{i,j=1}^{n}\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_j)\rangle \\ &= \frac{2}{n}\sum_{1\le i<j\le n}\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_j)\rangle + \frac{1}{n}\sum_{i=1}^{n}\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_i)\rangle, \\ &= \frac{2}{n}\sum_{1\le i<j\le n}\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_j)\rangle + M_n(M_n+2), \end{aligned} \tag{14}$$

where we used the fact that

$$\begin{aligned} \langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_i)\rangle &= \|t^{(1)}_{(M_n)}(\Theta_i)\|^2_{L^2} \\ &= \sum_{m=1}^{M_n}\langle t^{(1)}_{m}(\Theta_i), t^{(1)}_{m}(\Theta_i)\rangle \\ &= 2\sum_{m=1}^{M_n}\cos^2(m\Theta_i)+\sin^2(m\Theta_i) = 2M_n \end{aligned} \tag{15}$$

and that

$$\begin{aligned}
\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_i)\rangle &= \|t^{(2)}_{(M_n)}(\mathbf{U}_i)\|^2_{L^2} \\
&= \sum_{m=1}^{M_n} \langle t^{(2)}_m(\mathbf{U}_i), t^{(2)}_m(\mathbf{U}_i)\rangle \\
&= \sum_{m=1}^{M_n} (2m+1)P_m(\mathbf{U}_i'\mathbf{U}_i) = \sum_{m=1}^{M_n}(2m+1)P_m(1) = M_n(M_n+2).
\end{aligned} \tag{16}$$

As a direct consequence we have that

$$\begin{aligned}
S^{\text{stand}}_{1,M_n} &= \frac{S_{1,M_n} - 2M_n}{2\sqrt{M_n}} \\
&= \frac{1}{n\sqrt{M_n}} \sum_{1\le i<j\le n} \langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_j)\rangle \\
&= \frac{2}{n\sqrt{M_n}} \sum_{m=1}^{M_n} \sum_{1\le i<j\le n} \cos(m\Theta_i)\cos(m\Theta_j) + \sin(m\Theta_i)\sin(m\Theta_j)
\end{aligned} \tag{17}$$

and that

$$\begin{aligned}
S^{\text{stand}}_{2,M_n} &= \frac{S_{2,M_n} - M_n(M_n+2)}{\sqrt{2M_n(M_n+2)}} \\
&= \frac{\sqrt{2}}{n\sqrt{M_n(M_n+2)}} \sum_{1\le i<j\le n} \langle t_{(M_n)}(\mathbf{U}_i), t_{(M_n)}(\mathbf{U}_j)\rangle \\
&= \frac{\sqrt{2}}{n\sqrt{M_n(M_n+2)}} \sum_{1\le i<j\le n} \sum_{m=1}^{M_n} (2m+1)P_m(\mathbf{U}_i'\mathbf{U}_j).
\end{aligned} \tag{18}$$

We start now with the proof of Proposition 4.1.

Proof of Proposition 4.1. Our objective is to show that S^{stand}_{1,M_n} is asymptotically standard normal as $n \to \infty$ (and $M_n \to \infty$) under the null hypothesis of uniformity. First note that letting $\mathbf{Z}_{i,m} := (\cos(m\Theta_i), \sin(m\Theta_i))'$, we have that S^{stand}_{1,M_n} in (17) can be written as

$$S^{\text{stand}}_{1,M_n} := \frac{2}{n\sqrt{M_n}} \sum_{m=1}^{M_n} \sum_{1\le i<j\le n} \mathbf{Z}'_{i,m}\mathbf{Z}_{j,m} =: \frac{2}{n\sqrt{M_n}} \sum_{m=1}^{M_n} \sum_{1\le i<j\le n} \rho_{ij,m}.$$

Note that if Θ_i is uniform on $(0, 2\pi]$, clearly, $\mathbf{Z}_{i,m}$ is uniform on $\mathbb{S}^1$ and the appendix of Paindaveine and Verdebout (2016) summarizes a lot of properties of the $\rho_{ij,m}$'s in the uniform case.

We will now construct a martingale difference process. Define $\mathcal{F}_{n\ell}$ as the σ-algebra generated by $\Theta_1, \ldots, \Theta_\ell$, and, writing $\mathrm{E}_{n\ell}$ for the conditional expectation with respect to $\mathcal{F}_{n\ell}$, we let

$$\begin{aligned} S_{1\ell} &:= \mathrm{E}_{n\ell}\big[S_{1,M_n}^{\text{stand}}\big] - \mathrm{E}_{n,\ell-1}\big[S_{1,M_n}^{\text{stand}}\big] \\ &= \frac{2}{n\sqrt{M_n}} \sum_{m=1}^{M_n} \sum_{i=1}^{\ell-1} \rho_{i\ell,m} \\ &= \frac{1}{n\sqrt{M_n}} \sum_{i=1}^{\ell-1} \langle t_{(M_n)}^{(1)}(\Theta_i), t_{(M_n)}^{(1)}(\Theta_\ell)\rangle. \end{aligned}$$

We obviously have that $S_{1,M_n}^{\text{stand}} = \sum_{\ell=1}^n S_{1\ell}$. By construction, the process $S_{1\ell}$ is a martingale difference process so that in order to apply the Central limit Theorem 35.12 in Billingsley (1995) we need to show that

Lemma A.1. *Letting $\sigma_{n\ell}^2 = \mathrm{E}_{n,\ell-1}\big[(S_{1\ell})^2\big]$, $\sum_{\ell=1}^n \sigma_{n\ell}^2$ converges to one in probability as $n \to \infty$.*

together with the Lindeberg condition

Lemma A.2. *For any $\varepsilon > 0$, $\sum_{\ell=1}^n \mathrm{E}\big[(S_{1\ell})^2\, \mathbb{I}[|S_{1\ell}| > \varepsilon]\big] \to 0$ as $n \to \infty$.*

Proof of Lemma A.1. First note that

$$\begin{aligned} \sigma_{n\ell}^2 &= \mathrm{E}_{n,\ell-1}\big[(S_{1\ell})^2\big] \\ &= \frac{4}{n^2 M_n} \mathrm{E}_{n,\ell-1}\Big[\sum_{s,t=1}^{M_n} \sum_{i,j=1}^{\ell-1} \rho_{i\ell,s}\rho_{j\ell,t}\Big] \\ &= \frac{4}{n^2 M_n} \mathrm{E}_{n,\ell-1}\Big[\sum_{s,t=1}^{M_n} \sum_{i,j=1}^{\ell-1} \mathbf{Z}'_{i,s}\mathbf{Z}_{\ell,s}\mathbf{Z}'_{\ell,t}\mathbf{Z}_{j,t}\Big] \\ &= \frac{4}{n^2 M_n} \sum_{s,t=1}^{M_n} \sum_{i,j=1}^{\ell-1} \mathbf{Z}'_{i,s}\mathrm{E}[\mathbf{Z}_{\ell,s}\mathbf{Z}'_{\ell,t}]\mathbf{Z}_{j,t} \\ &= \frac{2}{n^2 M_n} \sum_{s=1}^{M_n} \sum_{i,j=1}^{\ell-1} \mathbf{Z}'_{i,s}\mathbf{Z}_{j,s} \\ &= \frac{2}{n^2 M_n} \sum_{s=1}^{M_n} \sum_{i,j=1}^{\ell-1} \rho_{ij,s} \\ &= \frac{1}{n^2 M_n} \sum_{i,j=1}^{\ell-1} \langle t_{(M_n)}^{(1)}(\Theta_i), t_{(M_n)}^{(1)}(\Theta_j)\rangle \end{aligned}$$

Since $\mathrm{E}[\rho_{ij,s}] = 0$ for $i \neq j$ and any s, it directly follows from (15) that

$$\mathrm{E}[\sigma^2_{n\ell}] \quad = \quad \frac{2(l-1)}{n^2}$$

so that

$$\sum_{\ell=1}^{n} \mathrm{E}[\sigma^2_{n\ell}] \quad = \quad \sum_{\ell=1}^{n} \frac{2(l-1)}{n^2} = 1.$$

Therefore, it remains to show that $\mathrm{Var}[\sum_{\ell=1}^{n} \sigma^2_{n\ell}]$ converges to zero as $n \to \infty$. Again from (15) we have that

$$\begin{aligned}
\mathrm{Var}[\sum_{\ell=1}^{n} \sigma^2_{n\ell}] \quad &= \quad \mathrm{Var}[\frac{1}{n^2 M_n} \sum_{\ell=1}^{n} \sum_{1 \leq i < j \leq (\ell-1)} \langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_j) \rangle] \\
&= \quad \frac{4}{n^4 M_n^2} \mathrm{Var}[\sum_{s=1}^{M_n} \sum_{1 \leq i < j \leq (n-1)} (n-j)\rho_{ij,s}] \\
&= \quad \frac{4}{n^4 M_n^2} \mathrm{E}[(\sum_{s=1}^{M_n} \sum_{1 \leq i < j \leq (n-1)} (n-j)\rho_{ij,s})^2] \qquad (19)
\end{aligned}$$

Since for $i < j$ and $k < \ell$, $\mathrm{E}[\rho_{ij,s}\rho_{k\ell,t}] = 0$ unless $(i,j) = (k,\ell)$ and $s = t$. In the latter case we have $\mathrm{E}[\rho^2_{ij,s}] = \frac{1}{2}$. It therefore follows from (19) that

$$\begin{aligned}
\mathrm{Var}[\sum_{\ell=1}^{n} \sigma^2_{n\ell}] \quad &= \quad \frac{4}{n^4 M_n^2} \sum_{s=1}^{M_n} \sum_{1 \leq i < j \leq (n-1)} (n-j)^2 \mathrm{E}[\rho^2_{ij,s}] \\
&\leq \quad \frac{2}{M_n}
\end{aligned}$$

which is $o(1)$ as $n \to \infty$. □

For the proof of Lemma A.2, we need the following preliminary result.

Lemma A.3. *Let Θ be a random angle uniformly distributed on $(0, 2\pi)$. Define*

$$I_1(a,b,c,d) = \mathrm{E}[\cos(a\Theta)\cos(b\Theta)\cos(c\Theta)\cos(d\Theta)],$$

$$I_2(a,b,c,d) := \mathrm{E}[\cos(a\Theta)\cos(b\Theta)\cos(c\Theta)\sin(d\Theta)],$$

$$I_3(a,b,c,d) := \mathrm{E}[\cos(a\Theta)\cos(b\Theta)\sin(c\Theta)\sin(d\Theta)],$$

$$I_4(a,b,c,d) := \mathrm{E}[\cos(a\Theta)\sin(b\Theta)\sin(c\Theta)\sin(d\Theta)]$$

and

$$I_5(a,b,c,d) := \mathrm{E}[\sin(a\Theta)\sin(b\Theta)\sin(c\Theta)\sin(d\Theta)].$$

Then we have that (a) $I_j(a,b,c,d) = 0,\ j = 2,4,$ *for all integers* $\{a,b,c,d\}$. *(b)* $I_j(a,b,c,d) = 0,\ j = 1,5,$ *unless any of the integers* $\{a,b,c,d\}$ *is equal to the sum of the other three (e.g,* $a = b+c+d$*), or if any two pairs of* $\{a,b,c,d\}$ *have equal sum (e.g,* $a+b = c+d$*). (c)* $I_3(a,b,c,d) = 0$, *unless unless any of the integers* $\{a,b,c,d\}$ *is equal to the sum of the other three, or if any two pairs of* $\{a,b,c,d\}$ *have equal sum, except for the cases* $\{a = d, b = c\}$ *and* $\{a = c, b = d\}$ *for which* $I_3(a,b,c,d) = 0$.

Proof of Lemma A.3. We show it for $I_1(a,b,c,d)$; the other cases are treated similarly. By simple trigonometric identities we have that

$$\begin{aligned}\int_0^{2\pi} \cos(a\vartheta)\cos(b\vartheta)\cos(c\vartheta)\cos(d\vartheta)d\vartheta \quad &= \quad \frac{\sin\left[2\pi(a-b-c-d)\right]}{2\pi(a-b-c-d)} + \frac{\sin\left[2\pi(a+b-c-d)\right]}{2\pi(a+b-c-d)} \\ &+ \frac{\sin\left[2\pi(a-b+c-d)\right]}{2\pi(a-b+c-d)} + \frac{\sin\left[2\pi(a+b+c-d)\right]}{2\pi(a+b+c-d)} + \frac{\sin\left[2\pi(a-b-c+d)\right]}{2\pi(a-b-c+d)} \\ &+ \frac{\sin\left[2\pi(a+b-c+d)\right]}{2\pi(a+b-c+d)} + \frac{\sin\left[2\pi(a-b+c+d)\right]}{2\pi(a-b+c+d)} + \frac{\sin\left[2\pi(a+b+c+d)\right]}{2\pi(a+b+c+d)},\end{aligned}$$

and the result follows immediately. □

Proof of Lemma A.2 Applying first the Cauchy-Schwarz inequality, then the Chebyshev inequality (note that $S^R_{n\ell}$ has zero mean), yields

$$\begin{aligned}\sum_{\ell=1}^{n} \mathrm{E}\left[(S_{1\ell})^2\, \mathbb{I}[|S_{1\ell}| > \varepsilon]\right] \quad &\leq \quad \sum_{\ell=1}^{n} \sqrt{\mathrm{E}\left[(S_{1\ell})^4\right]}\,\sqrt{\mathrm{P}[|S_{1\ell}| > \varepsilon]} \\ &\leq \quad \frac{1}{\varepsilon}\sum_{\ell=1}^{n} \sqrt{\mathrm{E}\left[(S_{1\ell})^4\right]}\,\sqrt{\mathrm{Var}[S_{1\ell}]}.\end{aligned}$$

From the proof of Lemma B1, we readily obtain that $\mathrm{Var}[S_{1\ell}] = 2(\ell-1)/n^2$, which provides

$$\sum_{\ell=1}^{n} \mathrm{E}\left[(S_{1\ell})^2\, \mathbb{I}[|S_{1\ell}| > \varepsilon]\right] \leq \frac{\sqrt{2}}{\varepsilon n}\sum_{\ell=1}^{n} \sqrt{(\ell-1)\mathrm{E}\left[(S_{1\ell})^4\right]}. \tag{20}$$

Now since

$$2\sum_{m=1}^{M_n} \rho_{i\ell,m} = \langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_\ell)\rangle,$$

we have using the properties of the $\rho_{i\ell,s}$'s that

$$\begin{aligned}
\mathrm{E}\big[(S_{1\ell})^4\big] &= \frac{16}{n^4 M_n^2} \sum_{s,t,u,v=1}^{M_n} \sum_{i=1}^{\ell-1} \mathrm{E}\big[\rho_{i\ell,s}\rho_{i\ell,t}\rho_{i\ell,u}\rho_{i\ell,v}\big] \\
&\quad + \frac{1}{n^4 M_n^2} \sum_{i,j=1}^{\ell-1} \mathrm{E}\big[\langle t^{(1)}_{(M_n)}(\Theta_i), t^{(1)}_{(M_n)}(\Theta_\ell)\rangle^2 \langle t^{(1)}_{(M_n)}(\Theta_j), t^{(1)}_{(M_n)}(\Theta_\ell)\rangle^2\big] \\
&= \frac{16}{n^4 M_n^2} \sum_{s,t,u,v=1}^{M_n} \sum_{i=1}^{\ell-1} \mathrm{E}\big[\rho_{i\ell,s}\rho_{i\ell,t}\rho_{i\ell,u}\rho_{i\ell,v}\big] + \frac{4(\ell-1)^2}{n^4}. \qquad (21)
\end{aligned}$$

In the sequel denote by $\mathcal{M}_3^{(n)}$ the set of integers $(a,b,c,d) \in (1,\ldots,M_n)^4$ such that any of the integers $\{a,b,c,d\}$ is equal to the sum of the other three and by $\mathcal{M}_2^{(n)}$ the set of integers $(a,b,c,d) \in (1,\ldots,M_n)^4$ such there exists two pairs in $\{a,b,c,d\}$ that have equal sum. Since from Section 3 in Knessl and Keller (1990), it follows that $\#(\mathcal{M}_2^{(n)} \Delta \mathcal{M}_3^{(n)}) = O(M_n^3)$ (Δ denotes the difference of two sets) as $M_n \to \infty$, the boundedness of the $\rho_{i\ell,s}$'s and Lemma A.3 entail that from (21) we have that

$$\begin{aligned}
\mathrm{E}\big[(S_{1\ell})^4\big] &= \frac{64}{n^4 M_n^2} \sum_{s,t,u,v=1}^{M_n} \sum_{i=1}^{\ell-1} \mathrm{E}\big[\rho_{i\ell,s}\rho_{i\ell,t}\rho_{i\ell,u}\rho_{i\ell,v}\big] + \frac{4(\ell-1)^2}{n^4} \\
&= \frac{64}{n^4 M_n^2} \sum_{(s,t,u,v)\in\mathcal{M}_2^{(n)}\Delta\mathcal{M}_3^{(n)}} \sum_{i=1}^{\ell-1} \mathrm{E}\big[\rho_{i\ell,s}\rho_{i\ell,t}\rho_{i\ell,u}\rho_{i\ell,v}\big] + \frac{4(\ell-1)^2}{n^4} \\
&\leq \frac{(\ell-1)C_n}{n^4 M_n^2} + + \frac{4(\ell-1)^2}{n^4},
\end{aligned}$$

where $C_n = O(M_n^3)$. Therefore, we obtain that for some constant C that

$$\begin{aligned}
\sum_{\ell=1}^{n} \mathrm{E}\big[(S_{1\ell})^2\, \mathbb{I}[|S_{1\ell}| > \varepsilon]\big] &\leq \frac{\sqrt{2}}{\varepsilon n} \sum_{\ell=1}^{n} \sqrt{(\ell-1)\mathrm{E}\big[(S_{1\ell})^4\big]} \\
&\leq \frac{\sqrt{2}}{\varepsilon n^3} \sum_{\ell=1}^{n} \sqrt{\frac{C_n(\ell-1)^2}{M_n^2} + 4(\ell-1)^3} \\
&\leq \frac{\sqrt{2}}{\varepsilon n^3} \sum_{\ell=1}^{n} \Big(\frac{C_n^{1/2}(\ell-1)}{M_n} + 4(\ell-1)^{3/2}\Big) = O(M_n^{1/2} n^{-1}),
\end{aligned}$$

which is $o(1)$ provided that $M_n = o(n^2)$.

□

We now move to the proof of Proposition 7.1.

Proof of Proposition 7.1. In this proof, we use the same techniques as in the proof of Proposition 4.1. We have that $S^{\rm stand}_{2,M_n}$ in (18) may be rewritten as

$$\begin{aligned} S^{\rm stand}_{2,M_n} &= \frac{\sqrt{2}}{n\sqrt{M_n(M_n+2)}}\sum_{\ell=1}^{n}\sum_{i=1}^{\ell-1}\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_\ell)\rangle \\ &=: \sum_{\ell=1}^{n} S_{2\ell}, \end{aligned} \tag{22}$$

where $S_{2\ell}$ is a martingale difference process. To obtain the asymptotic normality result we therefore need to obtain the following lemma.

Lemma A.4. *Letting $\sigma^2_{n\ell} = {\rm E}_{n,\ell-1}\big[(S_{2\ell})^2\big]$, $\sum_{\ell=1}^{n}\sigma^2_{n\ell}$ converges to one in probability as $n\to\infty$.*

together with the Lindeberg condition

Lemma A.5. *For any $\varepsilon>0$, $\sum_{\ell=1}^{n}{\rm E}\big[(S_{2\ell})^2\,\mathbb{I}[|S_{2\ell}|>\varepsilon]\big]\to 0$ as $n\to\infty$.*

The proof of Lemma A.4 follows along the same lines as the proof of Lemma A.1 using the fact that

$$\begin{aligned} \sigma^2_{n\ell} &=: {\rm E}_{n,\ell-1}\big[(S_{2\ell})^2\big] \\ &= \frac{2}{n^2M_n(M_n+2)}{\rm E}_{n,\ell-1}\Big[\sum_{i,j=1}^{\ell-1}\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_\ell)\rangle\langle t^{(2)}_{(M_n)}(\mathbf{U}_j), t^{(2)}_{(M_n)}(\mathbf{U}_\ell)\rangle\Big] \\ &= \frac{2}{n^2M_n(M_n+2)}\sum_{i,j=1}^{\ell-1}\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_j)\rangle \end{aligned} \tag{23}$$

has expectation $2(\ell-1)/n^2$ as in Lemma A.1. We directly turn to the proof of Lemma A.5.

Proof of Lemma A.5. Following the same lines as in the proof of Lemma A.2, we need to show that

$$\sum_{\ell=1}^{n}\sqrt{{\rm E}\big[(S_{2\ell})^4\big]}\,\sqrt{{\rm Var}[S_{2\ell}]}$$

is $o(1)$ as $n\to\infty$. First note that from (23), we easily obtain that

$${\rm Var}[S_{2\ell}] = {\rm E}[S^2_{2\ell}] = {\rm E}[\sigma^2_{n\ell}] = \frac{2(\ell-1)}{n^2}.$$

Now letting $\rho_{ij} := \langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_j)\rangle$ and using the properties of the inner products $\mathbf{U}_i'\mathbf{U}_j$ summarized in Lemma A1 of Paindaveine and Verdebout (2016) we have that

$$\begin{aligned}
\mathrm{E}\big[(S_{2\ell})^4\big] &= \frac{4}{n^4 M_n^2(M_n+2)^2}\sum_{i,j,k,m=1}^{\ell-1}\mathrm{E}[\rho_{i\ell}\rho_{j\ell}\rho_{k\ell}\rho_{m\ell}] \\
&= \frac{4}{n^4 M_n^2(M_n+2)^2}(\sum_{i=1}^{\ell-1}\mathrm{E}[\rho_{i\ell}^4] + \sum_{i,j=1}^{\ell-1}\mathrm{E}[\rho_{i\ell}^2\rho_{j\ell}^2]) \\
&= \frac{4}{n^4 M_n^2(M_n+2)^2}\sum_{i=1}^{\ell-1}\mathrm{E}[\rho_{i\ell}^4] + \frac{4(\ell-1)^2}{n^4}. \qquad (24)
\end{aligned}$$

Now we have that

$$\begin{aligned}
\mathrm{E}[\rho_{i\ell}^4] &= \mathrm{E}[\langle t^{(2)}_{(M_n)}(\mathbf{U}_i), t^{(2)}_{(M_n)}(\mathbf{U}_\ell)\rangle^4] \\
&= \sum_{s,t,u,v=1}^{M_n}(2s+1)(2t+1)(2u+1)(2v+1)\mathrm{E}[P_s(\mathbf{U}_i'\mathbf{U}_\ell)P_t(\mathbf{U}_i'\mathbf{U}_\ell)P_u(\mathbf{U}_i'\mathbf{U}_\ell)P_v(\mathbf{U}_i'\mathbf{U}_\ell)] \qquad (25)
\end{aligned}$$

It follows from Lemma A1 in Paindaveine and Verdebout (2016) that on $\mathbb{S}^2$, $\mathbf{U}_i'\mathbf{U}_j$ is uniformly distributed on $[-1,1]$. Using Theorem 2 in Lohöfer (1991), we have for $s \geq 1$ that $|P_s(x)| \leq \frac{\Gamma(1/4)}{\pi s^{1/4}(1-x^2)^{1/8}}$ for $x \in (-1,1)$. We therefore have from (25) that for some constant C

$$\mathrm{E}[\rho_{i\ell}^4] \leq C(\sum_{s=1}^{M_n}(2s+1)s^{-1/4})^4 =: C_n,$$

where $C_n = O(M_n^7)$. It follows from (24) that

$$\mathrm{E}\big[(S_{2\ell})^4\big] \leq \frac{4C_n(\ell-1)}{n^4 M_n^2(M_n+2)^2} + \frac{4(\ell-1)^2}{n^4}$$

and therefore that for some constant C

$$\begin{aligned}
\sum_{\ell=1}^{n}\sqrt{\mathrm{E}\big[(S_{2\ell})^4\big]}\sqrt{\mathrm{Var}[S_{2\ell}]} &\leq \frac{C}{n^3}\sum_{\ell=1}^{n}\sqrt{\frac{C_n(\ell-1)^2}{M_n^2(M_n+2)^2} + (\ell-1)^3} \\
&\leq \frac{C}{n^3}\sum_{\ell=1}^{n}(\frac{C_n^{1/2}(\ell-1)}{M_n(M_n+2)} + (\ell-1)^{3/2}) = O(M_n^{3/2}n^{-1}),
\end{aligned}$$

which is $o(1)$ as $n \to \infty$ provided that $M_n = o(n^{2/3})$ $\square$

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