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New Families of Wrapped Distributions for Modeling Skew Circular Data

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ABSTRACT

We discuss circular distributions obtained by wrapping the classical exponential and Laplace distributions on the real line around the circle. We present explicit forms for their densities and distribution functions, as well as their trigonometric moments and related parameters, and discuss main properties of these laws. Both distributions are very promising as models for asymmetric directional data.

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1. INTRODUCTION

Let X be a real random variable (r.v.) with probability density function (p.d.f.) f and characteristic function (ch.f.) ϕ. Then the corresponding wrapped r.v.

$$X_w = X(\text{mod } 2\pi) \tag{1.1}$$

has the density

$$f_w(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi), \quad \theta \in [0, 2\pi), \tag{1.2}$$

and the characteristic function (the discrete Fourier transform)

$$\phi_p = Ee^{ipX_w} = \phi(p), \quad p = 0, \pm 1, \pm 2, \ldots, \tag{1.3}$$

see, e.g., Fisher (1993), Jammalamadaka and SenGupta (2001) and Mardia and Jupp (2000).

The cases of wrapped Cauchy, normal, and stable distributions have been studied extensively (see, e.g., Gatto and Jammalamadaka, 2003; Levy, 1939). In this work, we present basic theory of *wrapped exponential and double-exponential* (Laplace) distributions, which were introduced recently in Jammalamadaka and Kozubowski (2001, 2003).

The exponential distribution, with its important memoryless property, provides a standard model in diverse fields such as reliability, queueing theory, and others (see, e.g., Barlow and Proschen, 1996). The classical Laplace distribution and its skew generalizations are competitors to normal and other symmetric distributions in stochastic modeling, particularly in financial applications (see, e.g., Kotz et al., 2001 and references therein). We believe that their circular analogs can find interesting applications in directional data, when they resemble the characteristic shape of the Laplace distribution (with its possible skewness and sharp peak at the mode). Such data frequently result from orientation experiments in biology (see, e.g., Batschelet, 1981; Bruderer, 1975; Jander, 1957; Matthews, 1961; Schmidt-Koenig, 1964). As an example of a typically skew distribution, Batschelet (1981) shows a histogram of directions chosen by migrating birds (see Fig. 2.6.1. in Batschelet, 1981). Commenting on the "asymmetry in the choice of directions", he remarked that it "may be caused by a mixture of two or three distributions". The wrapped Laplace distribution can in fact be represented as such a mixture.

Mardia (1972) presents several examples of skew empirical distributions, and states that "symmetrical distributions on the circle are comparatively rare" (see Mardia, 1972, p. 10). One of his examples is studied in detail by Pewsey (2002), as an illustration of his test of symmetry for circular data. Pewsey (2002) considered grouped data on the frequencies of thunder during various times of the day recorded at Kew (England) during the summers of 1910–1935 (see Table 1.3 in Mardia, 1972). Mardia (1972) classified these data as unimodal, "slightly asymmetrical" and "positively skew" (see Mardia, 1972, p. 10). When testing these data, Pewsey (2002) obtained the *p*-value of zero, rejecting the symmetry. Incidentally, when assessing the performance of his test of symmetry, Pewsey (2002) used wrapped (symmetric) Laplace distribution (as one of four symmetric distributions) and wrapped exponential distribution (as one of four skew alternatives) in his simulation study.

Below we sketch a theory of wrapped exponential and Laplace distributions. First, we define them in Sec. 2, where we also present their basic properties. Then, we summarize their important properties in Sec. 3. More detailed information with estimation, applications, and proofs can be found in Jammalamadaka and Kozubowski (2001, 2003).

2. DEFINITIONS AND BASIC PROPERTIES

When we apply (1.2) and (1.3) to the exponential distribution with p.d.f. $f(x) = \lambda e^{-\lambda x}$, $x > 0$, we obtain a wrapped exponential distribution, denoted by $WE(\lambda)$ with $\lambda > 0$. The following Table 1 provides

Table 1. The wrapped exponential distribution $WE(\lambda)$.

ch.f.	$\phi_p = \frac{1}{1-ip/\lambda}$,	$p = 0, \pm1, \pm2, \ldots$
p.d.f.	$f_w(\theta) = \frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}$,	$\theta \in [0, 2\pi)$
c.d.f.	$F_w(\theta) = \frac{1-e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}$,	$\theta \in [0, 2\pi)$

the characteristic function (ch.f.), density function (p.d.f.), and cumulative distribution function (c.d.f.) of the wrapped exponential distribution $WE(\lambda)$, $\lambda \in R$.

The p.d.f. should be extended in a periodic fashion for the values of θ outside of the interval $[0, 2\pi)$. The case $\lambda = 0$ is defined by a continuous extension and corresponds to the circular uniform distribution (when $\lambda \to 0^+$ then the c.d.f. in Table 1 converges to $\frac{\theta}{2\pi}$). The case $\lambda < 0$ results from wrapping the "negative" exponential distribution with parameter $|\lambda| > 0$, whose p.d.f. is $f(x) = |\lambda| e^{|\lambda| x}$, $x < 0$. Moreover, we have the relation

$$\Theta \sim WE(\lambda) \quad \text{if and only if } 2\pi - \Theta \sim WE(-\lambda). \tag{2.1}$$

Remark 2.1. We note a very interesting and curious property: The restriction of the linear exponential r.v. X to the interval $[0, 2\pi)$ (that is $X|X < 2\pi$) has the same distribution as the wrapped r.v. X_w given by (1.1).

Consider now an asymmetric Laplace r.v. X with the density function

$$f(x) = \lambda(1/\kappa + \kappa)^{-1} \begin{cases} e^{-\lambda\kappa|x|}, & \text{for } x \geq 0, \\ e^{-(\lambda/\kappa)|x|}, & \text{for } x < 0, \end{cases} \tag{2.2}$$

and the ch.f.

$$\begin{aligned} \phi(t) &= \frac{1}{1 + t^2/\lambda^2 - it(1/\kappa - \kappa)/\lambda} \\ &= \frac{1}{1 - it/(\lambda\kappa)} \frac{1}{1 + it/(\lambda/\kappa)}, \quad t \in R, \end{aligned} \tag{2.3}$$

where $\lambda, \kappa > 0$; see Kotz et al. (2001) for theory and applications of these distributions and their various generalizations. (When $\kappa = 1$ we obtain the classical (symmetric) Laplace density.) When we apply (1.2) and (1.3) to the above distribution, we obtain a *wrapped Laplace distribution*, denoted by $WL(\lambda, \kappa)$. This is defined in Table 2 below, which provides the ch.f., the p.d.f., and the c.d.f. of the wrapped Laplace distribution $WL(\lambda, \kappa)$, $\lambda \in R, \kappa > 0$.

The p.d.f. should be extended in a periodic fashion for the values of θ outside of the interval $[0, 2\pi)$. Observe that the p.d.f. in Table 2 is positive and integrates to 1 on $[0, 2\pi)$ for any value of λ, including $\lambda < 0$, and we have

$$WL(-\lambda, \kappa) = WL(\lambda, 1/\kappa). \tag{2.4}$$

Table 2. The wrapped Laplace distribution $WL(\lambda, \kappa)$.

ch.f.	$\phi_p = \frac{1}{1-ip/(\lambda\kappa)} \frac{1}{1+ip/(\lambda/\kappa)}$,	$p = 0, \pm 1, \pm 2, \ldots$
p.d.f.	$f_w(\theta) = \frac{\lambda\kappa}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\theta}}{1-e^{-2\pi\lambda\kappa}} + \frac{e^{(\lambda/\kappa)\theta}}{e^{2\pi\lambda/\kappa}-1}\right)$,	$\theta \in [0, 2\pi)$
c.d.f.	$F_w(\theta) = \frac{1}{1+\kappa^2}\frac{1-e^{-\lambda\kappa\theta}}{1-e^{-2\pi\lambda\kappa}} + \frac{\kappa^2}{1+\kappa^2}\frac{e^{(\lambda/\kappa)\theta}-1}{e^{2\pi\lambda/\kappa}-1}$,	$\theta \in [0, 2\pi)$

It is also easy to see that

$$\Theta \sim WL(\lambda, \kappa) \quad \text{if and only if} \quad 2\pi - \Theta \sim WL\left(\lambda, \frac{1}{\kappa}\right).$$

In case of a symmetric Laplace distribution with $\kappa = 1$, the wrapped Laplace density simplifies to

$$f_w(\theta) = \frac{\lambda}{2}\frac{e^{(2\pi-\theta)\lambda} + e^{\lambda\theta}}{e^{2\pi\lambda} - 1}, \tag{2.5}$$

and we have $2\pi - \Theta \stackrel{d}{=} \Theta$ in this case, where $\stackrel{d}{=}$ denotes distributional equivalence.

Remark 2.2. The Laplace distribution is a mixture of a positive and a negative exponential distributions, since f in (2.2) can be written as

$$f(x) = pf_1(x) + (1-p)f_2(x), \quad x \in R, \tag{2.6}$$

$$f_1(x) = \lambda_1 e^{-\lambda_1 x}\ (x > 0), \quad f_2(x) = \lambda_2 e^{\lambda_2 x}\ (x < 0), \tag{2.7}$$

and

$$p = \frac{1}{\kappa^2 + 1}, \quad \lambda_1 = \lambda\kappa, \quad \lambda_2 = \lambda/\kappa. \tag{2.8}$$

Consequently, the wrapped Laplace distribution is a mixture of two wrapped exponential distributions, as the p.d.f. in Table 2 can be written as

$$f_w(\theta) = p\frac{\lambda_1 e^{-\lambda_1\theta}}{1 - e^{-2\pi\lambda_1}} + (1-p)\frac{\lambda_2 e^{\lambda_2\theta}}{e^{2\pi\lambda_2} - 1}, \quad \theta \in [0, 2\pi), \tag{2.9}$$

with the parameters as in (2.8). The corresponding wrapped Laplace $WL(\lambda, \kappa)$ r.v. Θ admits the representation

$$\Theta \stackrel{d}{=} I\Theta_1 + (1 - I)\Theta_2, \tag{2.10}$$

where Θ_1 and Θ_2 are independent wrapped exponential $WE(\lambda\kappa)$ and $WE(-\lambda/\kappa)$ r.v.'s, and I is an indicator random variable (independent of Θ_1 and Θ_2) taking on the values 1 and 0 with probabilities $1/(1+\kappa^2)$ and $\kappa^2/(1+\kappa^2)$, respectively.

Remark 2.3. By the factorization of the asymmetric Laplace ch.f., the corresponding r.v. has the same distribution as the difference of two independent exponential random variables (see, e.g., Kotz et al., 2001). Since the wrapped Laplace ch.f. given in Table 2 admits a similar factorization, we obtain an analogous representation for the wrapped Laplace r.v. $\Theta \sim WL(\lambda, \kappa)$ viz.

$$\Theta \stackrel{d}{=} \Theta_1 + \Theta_2 \ (\mathrm{mod}\, 2\pi), \tag{2.11}$$

where Θ_1 and Θ_2 are independent wrapped exponential $WE(\lambda\kappa)$ and $WE(-\lambda/\kappa)$ r.v.'s, mentioned before.

Remark 2.4. The $WL(\lambda, \kappa)$ distribution converges weakly to the circular uniform distribution as $\lambda \to 0$, or $\kappa \to 0^+$, or $\kappa \to \infty$.

Remark 2.5. A three-parameter class of distributions can be defined by introducing a location parameter $\eta \in [0, 2\pi)$ and shifting the wrapped Laplace p.d.f. (defined on R by a periodic extension) by η with the resulting densities of the form

$$g(\theta) = f_w(\theta - \eta)$$

with f_w given in Table 2. Parameter η clearly corresponds to the mode for such a family.

2.1. Characterization of the Densities

Note that the p.d.f. of the wrapped exponential distribution $WE(\lambda)$ is strictly decreasing on the interval $[0, 2\pi)$ if $\lambda > 0$, and strictly increasing on $[0, 2\pi)$ if $\lambda < 0$.

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In case of the wrapped Laplace distributions, the densities are unimodal with a sharp peak at the mode. Basic properties of the densities are described in the following result taken from Jammalamadaka and Kozubowski (2003).

Proposition 2.1. *Let $f(\cdot;\lambda,\kappa)$ be the density of the $WL(\lambda,\kappa)$ distribution with $\lambda > 0$. Then*

(i) *$f(\cdot;\lambda,\kappa)$ is strictly decreasing on $(0,\theta^*)$ and strictly increasing on $(\theta^*, 2\pi)$, where*

$$\theta^* = \left(\frac{\lambda}{\kappa} + \lambda\kappa\right)^{-1} \ln\left(\kappa^2 \frac{e^{2\pi\lambda/\kappa} - 1}{1 - e^{-2\pi\lambda\kappa}}\right). \tag{2.12}$$

Moreover

$$\theta^* > \pi \;\; for\; \kappa < 1, \quad \theta^* = \pi \;\; for\; \kappa = 1, \quad and \quad \theta^* < \pi \;\; for\; \kappa > 1. \tag{2.13}$$

(ii) *The maximum and the minimum values of $f(\cdot;\lambda,\kappa)$ are*

$$f(0;\lambda,\kappa) = \lim_{\theta\to 2\pi} f(\theta;\lambda,\kappa) = \frac{\lambda\kappa}{1+\kappa^2}\left(\frac{1}{e^{2\pi\lambda/\kappa} - 1} + \frac{1}{1 - e^{-2\pi\lambda\kappa}}\right) \tag{2.14}$$

and

$$f(\theta^*;\lambda,\kappa) = e^{\frac{2\pi\lambda\kappa}{1+\kappa^2}} \left(\frac{e^{2\pi\lambda/\kappa} - 1}{2\pi\lambda/\kappa}\right)^{\frac{-\kappa^2}{1+\kappa^2}} \left(\frac{e^{2\pi\lambda\kappa} - 1}{2\pi\lambda\kappa}\right)^{\frac{-1}{1+\kappa^2}}, \tag{2.15}$$

respectively.

(iii) *For any given $\lambda \ge 0$ and $\kappa \ge 0$, we have*

$$f(\theta;\lambda,1/\kappa) = f(2\pi - \theta;\lambda,\kappa), \quad \theta \in [0, 2\pi). \tag{2.16}$$

2.2. Trigonometric Moments and Related Parameters

Computation of the trigonometric moments and related parameters of the wrapped exponential distribution is straightforward. For the convenience of the reader we summarize some common parameters in

Table 3. Note that since the ϕ_p's are the Fourier coefficients, we have the following Fourier representation of the wrapped exponential density:

$$f(\theta) = \sum_p \phi_p e^{-ip\theta} = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty} \frac{\lambda^2}{\lambda^2 + p^2}\cos p\theta + \frac{\lambda p}{\lambda^2 + p^2}\sin p\theta\right]. \tag{2.17}$$

Similarly the trigonometric moments of the wrapped Laplace distribution are also easy to derive, utilizing the mixture representation (2.10) and formulas for the trigonometric moments of the wrapped exponential distribution given in Table 3. Here, the density of the $WL(\lambda, \kappa)$ distribution admits the Fourier representation

$$f(\theta; \lambda, \kappa) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty} \frac{\kappa\lambda[\kappa\lambda(p^2 + \lambda^2)\cos p\theta + p\lambda^2(1 - \kappa^2)\sin p\theta]}{(\lambda^2\kappa^2 + p^2)(\kappa^2 p^2 + \lambda^2)}\right]. \tag{2.18}$$

Other common parameters of the wrapped Laplace laws can be obtained with ease, perhaps with the exception of the circular median. They are summarized in Table 4. Note that the mean direction lies in the interval $[0, \pi/2)$ for $\kappa \leq 1$ and in the interval $[3\pi/2, 2\pi)$ for $\kappa > 1$ (this restriction is caused by the fact that the mode is equal to 0).

A median direction ξ_0 of a circular distribution with density f is any solution (in the interval $[0, 2\pi)$) of

$$\int_{\xi_0}^{\xi_0+\pi} f(\theta)d\theta = \int_{\xi_0+\pi}^{\xi_0+2\pi} f(\theta)d\theta = \frac{1}{2}, \tag{2.19}$$

where the density f satisfies

$$f(\xi_0) > f(\xi_0 + \pi), \tag{2.20}$$

see, e.g., Mardia and Jupp (2000). Clearly, the median direction of the wrapped symmetric Laplace distribution $WL(\lambda, 1)$ is equal to zero (and coincides with the mean direction and the mode). In the following result, proved in Jammalamadaka and Kozubowski (2003), we describe the median direction for the general case.

Table 3. Trigonometric moments and related parameters of the wrapped exponential distribution $WE(\lambda)$.

Parameter	Definition	Value
Trignometric moments	$\phi_p = \alpha_p + i\beta_p$	$\alpha_p = \frac{\lambda^2}{\lambda^2+p^2}$
	$p = 0, \pm 1, \pm 2, \ldots$	$\beta_p = \frac{p\lambda}{\lambda^2+p^2}$
Central trigonometric moments	$\bar{\alpha}_p = \rho_p \cos(\mu_p^0 - p\mu_0)$	$\bar{\alpha}_p = \frac{\vert\lambda\vert}{\sqrt{\lambda^2+p^2}} \cos\left(\tan^{-1}\frac{p}{\lambda} - p\tan^{-1}\frac{1}{\lambda}\right)$
	$\bar{\beta}_p = \rho_p \sin(\mu_p^0 - p\mu_0)$	$\bar{\beta}_p = \frac{\vert\lambda\vert}{\sqrt{\lambda^2+p^2}} \sin\left(\tan^{-1}\frac{p}{\lambda} - p\tan^{-1}\frac{1}{\lambda}\right)$
ρ_p and μ_p^0	$\phi_p = \rho_p e^{i\mu_p^0}, \quad p \geq 0,$	$\rho_p = \frac{\vert\lambda\vert}{\sqrt{\lambda^2+p^2}}$
		$\mu_p^0 = \begin{cases} \tan^{-1}\frac{p}{\lambda}, & \lambda > 0 \\ 2\pi + \tan^{-1}\frac{p}{\lambda}, & \lambda < 0 \end{cases}$
Resultant length	$\rho = \rho_1$	$\frac{\vert\lambda\vert}{\sqrt{\lambda^2+1}}$
Mean direction	$\mu_0 = \mu_1^0$	$\tan^{-1}\frac{1}{\lambda}, \quad \lambda > 0$ $2\pi + \tan^{-1}\frac{1}{\lambda}, \quad \lambda < 0$
Circular variance	$V_0 = 1 - \rho$	$\frac{\sqrt{1+\lambda^2} - \vert\lambda\vert}{\sqrt{1+\lambda^2}}$
Circular standard deviation	$\sigma_0 = \sqrt{-2\ln(1-V_0)}$	$\sqrt{\ln(1+1/\lambda^2)}$
Skewness	$\gamma_1^0 = \bar{\beta}_2 / V_0^{3/2}$	$\frac{-2\lambda}{(1+\lambda^2)^{1/4}(4+\lambda^2)(\sqrt{1+\lambda^2}-\lambda)^{3/2}}$
Kurtosis	$\gamma_2^0 = \frac{\bar{\alpha}_2 - (1-V_0)^4}{V_0^2}$	$\frac{3\lambda^2}{(1+\lambda^2)(4+\lambda^2)(\sqrt{1+\lambda^2}-\vert\lambda\vert)^2}$
Median	See (2.19)–(2.20)	$\frac{1}{\lambda}\ln\frac{2}{1+e^{-\lambda\pi}} + \begin{cases} 0, & \lambda > 0 \\ \pi, & \lambda < 0 \end{cases}$

Table 4. Trigonometric moments and related parameters of the wrapped Laplace distribution $WL(\lambda, \kappa)$.

Parameter(s)	Value(s)	Case $\kappa = 1$
Trigonometric moments	$\alpha_p = \frac{\kappa^2\lambda^2(p^2+\lambda^2)}{(\lambda^2\kappa^2+p^2)(p^2\kappa^2+\lambda^2)}$	$\alpha_p = \frac{\lambda^2}{p^2+\lambda^2}$
	$\beta_p = \frac{p\kappa\lambda^3(1-\kappa^2)}{(\lambda^2\kappa^2+p^2)(p^2\kappa^2+\lambda^2)}$	$\beta_p = 0$
Central trigonometric moments	$\bar{\alpha}_p = \rho_p \cos(\mu_p^0 - p\mu_0)$	$\bar{\alpha}_p = \frac{\lambda^2}{\lambda^2+p^2}$
	$\bar{\beta}_p = \rho_p \sin(\mu_p^0 - p\mu_0)$	$\bar{\beta}_p = 0$
ρ_p and μ_p^0	$\rho_p = \frac{\lambda^2}{\sqrt{\lambda^2\kappa^2+p^2}\sqrt{\lambda^2/\kappa^2+p^2}}, \quad p \geq 0$	$\rho_p = \frac{\lambda^2}{\lambda^2+p^2}$
	$\mu_p^0 = \begin{cases} \tan^{-1}\frac{p}{\lambda\kappa} - \tan^{-1}\frac{p\kappa}{\lambda}, & \kappa \leq 1 \\ 2\pi + \tan^{-1}\frac{p}{\lambda\kappa} - \tan^{-1}\frac{p\kappa}{\lambda}, & \kappa > 1 \end{cases}$	$\mu_p^0 = 0$
Resultant length	$\rho = \frac{\lambda^2}{\sqrt{1+\lambda^2\kappa^2}\sqrt{\lambda^2/\kappa^2+1}}$	$\rho = \frac{\lambda^2}{\lambda^2+1}$
Mean direction	$\mu_0 = \begin{cases} \tan^{-1}\frac{1}{\lambda\kappa} - \tan^{-1}\frac{\kappa}{\lambda}, & \kappa \leq 1 \\ 2\pi + \tan^{-1}\frac{1}{\lambda\kappa} - \tan^{-1}\frac{\kappa}{\lambda}, & \kappa > 1 \end{cases}$	$\mu_0 = 0$

Circular variance	$V_0 = 1 - \frac{\lambda^2}{\sqrt{1+\lambda^2\kappa^2}\sqrt{1+\lambda^2/\kappa^2}}$	$V_0 = \frac{1}{1+\lambda^2}$
Circular standard deviation	$\sigma_0 = \sqrt{\ln\left(\kappa^2 + \frac{1}{\lambda^2}\right) + \ln\left(\frac{1}{\kappa^2} + \frac{1}{\lambda^2}\right)}$	$\sigma_0 = \sqrt{2\ln\left(1 + \frac{1}{\lambda^2}\right)}$
Skewness	$\gamma_1^0 = \bar{\beta}_2 / V_0^{3/2}$, where	$\gamma_1^0 = 0$
	$\bar{\beta}_2 = \frac{2\lambda^3\kappa(\lambda^2\kappa^2+3) - 2(\lambda^3/\kappa)(\lambda^2/\kappa^2+3)}{(1+\lambda^2\kappa^2)(4+\lambda^2\kappa^2)(1+\lambda^2/\kappa^2)(4+\lambda^2/\kappa^2)}$	
Kurtosis	$\gamma_2^0 = \frac{\bar{\alpha}_2 - (1-V_0)^4}{V_0^2}$, where	$\gamma_2^0 = (1+\lambda^2)\lambda^2$
	$\hat{\alpha}_2 = \frac{\lambda^4(\lambda^2\kappa^2+3)(\lambda^2/\kappa^2+3) + 4\lambda^4}{(1+\lambda^2\kappa^2)(4+\lambda^2\kappa^2)(1+\lambda^2/\kappa^2)(4+\lambda^2/\kappa^2)}$	$\times \left(\frac{1}{4+\lambda^2} - \frac{\lambda^6}{(1+\lambda^2)^2}\right)$
Median	$\xi_0 = \begin{cases} \xi^*, & \lambda > 0,\ 0 < \kappa < 1 \\ \xi^* + \pi, & \lambda > 0,\ \kappa > 1, \text{ where} \end{cases}$	$\xi_0 = 0$
	$\frac{1}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\xi^*}}{1+e^{-\lambda\kappa\pi}} + \frac{\kappa^2 e^{\lambda\xi^*/\kappa}}{1+e^{\lambda\pi/\kappa}}\right) = \frac{1}{2}, \quad \xi^* \in [0, \pi]$	

Note: For the definitions see Table 3. The entries in the last column correspond to the symmetric case with $\kappa = 1$.

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Proposition 2.2. *Let $\Theta \sim WL(\lambda, \kappa)$. Then, Θ admits a unique median direction given by*

$$\xi_0 = \begin{cases} \xi^*, & \text{for } \lambda > 0, \ \ 0 < \kappa < 1, \\ \xi^* + \pi, & \text{for } \lambda > 0, \ \ \kappa > 1, \end{cases} \tag{2.21}$$

where $\xi^* \in [0, \pi]$ is the unique solution of the equation

$$\frac{1}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\xi}}{1+e^{-\lambda\kappa\pi}} + \kappa^2 \frac{e^{\lambda\xi/\kappa}}{1+e^{\lambda\pi/\kappa}}\right) = \frac{1}{2}. \tag{2.22}$$

Clearly, the median is less than π for $\kappa < 1$ and greater than π for $\kappa > 1$.

3. SOME IMPORTANT PROPERTIES

In this section we show that wrapped exponential and Laplace distributions share some of the well-known properties of their analogues on the real line.

3.1. Infinite Divisibility

An angular r.v. Θ (and its probability distribution) is infinitely divisible if for any integer $n \geq 1$ there exist i.i.d. angular r.v.'s $\Theta_1, \ldots, \Theta_n$ such that

$$\Theta_1 + \cdots + \Theta_n (\text{mod}\, 2\pi) \stackrel{d}{=} \Theta. \tag{3.1}$$

Since a circular variable obtained by wrapping an infinitely divisible random variable is infinitely divisible (see Mardia and Jupp, 2000), the infinite divisibility of the wrapped exponential and Laplace distributions follow from that of the linear exponential and Laplace distributions.

Proposition 3.1. *If $\Theta \sim WE(\lambda)$ with $\lambda \in R$, then Θ is infinitely divisible. Moreover, for any positive integer $n \geq 1$ the equality in distribution* (3.1) *holds where the Θ_i's have the uniform circular distribution for $\lambda = 0$ and the wrapped gamma distribution with the ch.f.*

$$\phi_p = \left(\frac{1}{1 - ip/\lambda}\right)^{1/n} \tag{3.2}$$

for $\lambda \neq 0$.

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Proposition 3.2. *If* $\Theta \sim WL(\lambda, \kappa)$, *where* $\lambda, \kappa \geq 0$, *then* Θ *is infinitely divisible. Moreover, for any positive integer* $n \geq 1$ *the equality in distribution* (3.1) *holds with uniform circular variable* Θ_1 *if either* $\lambda = 0$ *or* $\kappa = 0$, *and otherwise with*

$$\Theta_1 \stackrel{d}{=} \Theta' + \Theta'' \,(\mathrm{mod}\, 2\pi), \tag{3.3}$$

where Θ' *and* Θ'' *are independent wrapped gamma r.v.'s with ch.f.'s*

$$\left(\frac{1}{1 - ip/(\lambda\kappa)}\right)^{1/n} \quad \textit{and} \quad \left(\frac{1}{1 + ip/(\lambda/\kappa)}\right)^{1/n}, \tag{3.4}$$

respectively.

3.2. Geometric Infinite Divisibility

Motivated by the stability property of the exponential distribution with respect to geometric compounding, Jammalamadaka and Kozubowski (2001) introduced a notion of geometric infinite divisibility for angular distributions.

Definition 3.1. An angular r.v. Θ is said to be geometric infinitely divisible if for any $q \in (0, 1)$ there exist i.i.d. angular r.v.'s $\Theta_1, \Theta_2, \ldots$ such that

$$\Theta_1 + \cdots + \Theta_{\nu_q} (\mathrm{mod}\, 2\pi) \stackrel{d}{=} \Theta, \tag{3.5}$$

where ν_q has the geometric distribution

$$P(\nu_q = k) = (1 - q)^{k-1} q, \quad k = 1, 2, 3, \ldots, \tag{3.6}$$

The following properties of wrapped exponential and Laplace distributions, established in Jammalamadaka and Kozubowski (2001, 2003), follow from analogous results of classical exponential and Laplace laws.

Proposition 3.3. *If* $\Theta \sim WE(\lambda)$, *where* $\lambda \in R$, *then* Θ *is geometric infinitely divisible. Moreover, for any* $q \in (0, 1)$ *the equality in distribution* (3.5) *holds, where the* Θ_i*'s have the uniform circular distribution for* $\lambda = 0$ *and the* $WE(\lambda/q)$ *distribution for* $\lambda \neq 0$.

Proposition 3.4. *If* $\Theta \sim WL(\lambda, \kappa)$, *where* $\lambda, \kappa \geq 0$, *then* Θ *is geometric infinitely divisible. Moreover, for any* $q \in (0, 1)$ *the equality* *in distribution* (3.5) *holds, where the* Θ_i*'s have the uniform circular distribution for* $\lambda = 0$ *or* $\kappa = 0$, *and the* $WL(\lambda_q, \kappa_q)$ *distribution for* $\lambda, \kappa > 0$. *Here,* $\lambda_q = \lambda/\sqrt{q}$ *and* κ_q *is the unique solution of the equation*

$$\frac{1}{\kappa_q} - \kappa_q = \sqrt{q}\left(\frac{1}{\kappa} - \kappa\right). \tag{3.7}$$

3.3. Maximum Entropy Property

Recall that the *entropy* of a r.v. Θ with p.d.f. f is defined as

$$H(\Theta) = -\int_0^{2\pi} f(\theta) \ln f(\theta) d\theta, \tag{3.8}$$

and provides a measure of uncertainty. Jaynes (1957) proposed general inference procedures of finding a distribution that maximizes the entropy, and this method has been applied in a variety of fields including statistical mechanics, stock-market analysis, queuing theory, and reliability (see, e.g., Kapur, 1993). It is well-known that if the mean direction and circular variance are fixed, then the entropy is maximized by the von Mises distribution, and under no restrictions on f the entropy is maximal for the circular uniform distribution (see, e.g., Kapur, 1993). As shown in Jammalamadaka and Kozubowski (2001), the wrapped exponential distribution $WE(\lambda)$ maximizes the entropy under the condition

$$\int_0^{2\pi} \theta f(\theta) d\theta = m, \quad 0 < m < 2\pi. \tag{3.9}$$

Proposition 3.5. *Consider the class C of all circular r.v.'s with density f satisfying the condition* (3.9). *Then, the maximum entropy is attained by the* $WE(\lambda)$ *distribution, where* $\lambda = (2\pi\xi)^{-1}$ *and* ξ *satisfies the equation*

$$\frac{m}{2\pi} = \xi - \frac{1}{e^{1/\xi} - 1}. \tag{3.10}$$

Moreover, the maximal entropy is

$$\max_{\Theta \in \mathscr{C}} H(\Theta) = \ln\left(\frac{1 - e^{-2\pi\lambda}}{\lambda}\right) + \frac{1}{\lambda} - \frac{2\pi e^{-2\pi\lambda}}{1 - e^{-2\pi\lambda}}. \tag{3.11}$$

Remark 3.1. By the monotonicity properties of the function $g(\xi) = \xi - (e^{1/\xi} - 1)^{-1}$, it follows that the distribution maximizing the entropy under the restriction (3.9) is wrapped exponential ($\lambda > 0$) for $0 < m < \pi$, wrapped negative exponential ($\lambda < 0$) for $\pi < m < 2\pi$, and circular uniform ($\lambda = 0$) for $m = \pi$.

Remark 3.2. This above result is analogous to the well-known property of the exponential distribution (which maximizes the entropy among all continuous probability distributions on $(0, \infty)$ with a given mean, see, e.g., Kapur, 1993).

SUMMARY

In this paper, we discuss circular distributions resulting from wrapping the exponential and Laplace distributions on the real line around the circle. The densities and distribution functions of wrapped exponential distributions admit explicit forms, as do trigonometric moments and related parameters. Wrapped exponential distributions retain the important properties of infinite divisibility and maxim entropy of the corresponding exponential distributions. The mixture of two wrapped exponential distributions leads to a wrapped Laplace distribution, so that the properties of the former distribution are useful in studying the latter one. Both distributions allow for skewness, and are promising for modeling asymmetric directional data.

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